

# A NEW WAY TO DETECT PAIRS OF NON-COBORDANT SURFACE-LINKS WHICH ORR INVARIANT, COCHRAN SEQUENCE, SATO-LEVINE INVARIANT, AND ALINKING NUMBER CANNOT DISTINGUISH

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ABSTRACT. We submit a new way to detect pairs of non-cobordant surface-links. We find a new example of a pair of non-cobordant surface-links with the following properties: Orr invariant, Cochran sequence, Sato-Levine invariant, the alinking number and one of Stallings's theorems cannot distinguish them. However our new way can distinguish them.

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## 1. INTRODUCTION AND THE MAIN RESULT

We work in the smooth category. Let  $\mu \in \mathbb{N}$ . Let  $i \in \{1, \dots, \mu\}$ . Let  $F_i$  be a connected closed oriented surface. A *surface- $(F_1, \dots, F_\mu)$ -link*, or  $(F_1, \dots, F_\mu)$ -*link*, is a submanifold  $L = (K_1, \dots, K_\mu)$  of  $S^4$  such that  $K_i$  is diffeomorphic to  $F_i$  for each  $i$ .

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In this paper, when we use the term, surface- $(F_1, \dots, F_\mu)$ -link, we suppose that each  $F_*$  is a connected closed oriented surface unless otherwise specified. When we use the term, a surface-link  $(L_1, \dots, L_\mu)$  or say that  $(L_1, \dots, L_\mu)$  is a surface-link, we suppose that each  $L_*$  is a connected closed oriented surface unless otherwise specified.

If all  $F_i$  are spheres,  $L$  is called a *2-dimensional spherical link*, or *spherical link*. If  $\mu = 1$ ,  $L$  is called a *surface- $F_1$ -knot*, or  *$F_1$ -knot*. (*2-links* mean spherical links in some cases and surface-links in the other cases.) A *Seifert hypersurface* for  $L$  is a connected compact oriented 3-dimensional submanifold  $V \subset S^4$  such that  $\partial V = L$ .

**Definition 1.1.** Surface- $(F_1, \dots, F_\mu)$ -links,  $L_0 = (K_{01}, \dots, K_{0\mu})$  and  $L_1 = (K_{11}, \dots, K_{1\mu})$ , are said to be *surface-link-cobordant*, *link-cobordant*, or *cobordant*, if there is an embedding map

$$f : (F_1 \times [0, 1]) \amalg \dots \amalg (F_\mu \times [0, 1]) \hookrightarrow S^4 \times [0, 1]$$

with the following properties. Let  $j \in \{0, 1\}$ . Let  $\mu \in \mathbb{N}$ . Let  $i \in \{1, \dots, \mu\}$ .

(1)  $f$  meets  $S^4 \times \{j\}$  transversely.

$$f^{-1}(S^4 \times \{j\}) = (F_1 \times \{j\}) \amalg \dots \amalg (F_\mu \times \{j\}).$$

(2)  $f((F_1 \times \{j\}) \amalg \dots \amalg (F_\mu \times \{j\}))$  (resp.  $f(F_i \times \{j\})$ ) in  $S^4 \times \{j\}$  is  $L_j$  (resp.  $K_{ji}$ ).

We say that an embedding map  $f$  (resp. a submanifold  $f((F_1 \times [0, 1]) \amalg \dots \amalg (F_\mu \times [0, 1])) \subset S^4 \times [0, 1]$ ) gives *cobordism* between  $L_0$  and  $L_1$ .

A *genus  $g$  handlebody* is a 3-dimensional compact connected oriented 3-manifold which consists of one 0-handle and  $g$  copies of 1-handle. Let  $\mu \in \mathbb{N}$ . Let  $i \in \{1, \dots, \mu\}$ . Let  $g_i$  be the genus of a closed oriented connected surface  $F_i$ . A surface- $(F_1, \dots, F_\mu)$ -link  $L = (K_1, \dots, K_\mu)$  is called the *standard link* if there is a disjoint embedded 3-dimensional submanifold  $(V_1, \dots, V_\mu) \subset S^4$  such that  $V_i$  is a genus  $g_i$  handlebody and such that for each  $i \in \{1, \dots, \mu\}$ ,  $V_i$  is a Seifert hypersurface for  $K_i$ . A surface- $(F_1, \dots, F_\mu)$ -link  $L$  is said to be *trivial* if  $L$  is spherical and standard. If a spherical link  $L$  is cobordant to the trivial link,  $L$  is said to be *slice*.

A surface- $(F_1, \dots, F_n)$ -link  $L = (K_1, \dots, K_n)$  is called a *boundary link* if there is a disjoint embedded 3-dimensional submanifold  $V_1 \amalg \dots \amalg V_n \subset S^4$ , where  $\amalg$  denotes the disjoint union, such that for each  $i \in \{1, \dots, n\}$ ,  $V_i$  is a Seifert hypersurface for  $K_i$ .

Let  $L = (K_1, \dots, K_n)$  be a surface- $(F_1, \dots, F_n)$ -link  $\subset S^4$ . Remove  $K_{j_1}, \dots, K_{j_p}$  from  $L$ . Suppose that the left components are  $K_{l_1}, \dots, K_{l_{n-p}}$ . We do not suppose that  $l_1 \leq \dots \leq l_{n-p}$ . We call the surface-link  $(K_{l_1}, \dots, K_{l_{n-p}})$  a *sublink* of  $L$ . We do not say that the empty set (resp.  $L$  itself) is a sublink of  $L$  in this paper.

**Problem 1.2.** Are all 2-dimensional spherical links slice?

This is a well-known outstanding open problem. See [5, 9, 11, 12, 14] for the history and the background. In order to attack this open problem in the future, in this paper we consider the following problem which includes the above one.

**Problem 1.3.** Classify surface-links up to surface-link-cobordism.

Why we consider Problem 1.3 together with Problem 1.2 is because we know many non-cobordant pairs in the case of all non-spherical surface-links ([4, 13, 14, 15, 16]). We will review an outline of the results in the following paragraphs. We hope the following: we shall continue to make new pairs of non-cobordant, non-spherical surface-links, and then we may solve Problem 1.2 in the future. Of course it itself is important to make such pairs. Our results, Main Theorem 1.4 and Theorem 9.2, show new examples which give answers to Problems 1.3.

Levine (unpublished), and Sato in [16] defined the Sato-Levine invariant which is a surface-link-cobordism invariant and which is trivial for the standard link. [15, 16] showed a  $(S^2, T^2)$ -link whose Sato-Levine invariant is nontrivial. See §3.

In [4] Cochran defined Cochran sequence which is a surface-link-cobordism invariant and which is trivial for the standard link. He also proved that Cochran sequence is nontrivial for a  $(S^2, T^2)$ -link. See §4.

In [14] Orr defined Orr invariant which is a surface-link-cobordism invariant under a condition and which is trivial for the standard link. [14, §5] claimed that Orr invariant is nontrivial for a surface-link. See §5. In §6 we prove a new property of Orr invariant (Theorem 6.5).

[16, §2] proved that if two surface-links are cobordant and the alinking number of one of the two is zero, then that of the other is zero. In [13, Proposition 7.10] the author generalized it and proved that if two surface-links are cobordant then the alinking number of the two are the same. Note that the former result does not imply the latter one directly. It is well-known that the alinking number is nontrivial for a  $(S^2, T^2)$ -link and that it is trivial for the standard link. See §2.

In [17, 5.2 Theorem] Stallings proved the following: For a surface-link  $A$ , let  $\pi_A$  be  $\pi_1(S^4 - A) \cong \pi_1(S^4 - N(A)) \cong \pi_1(S^4 - \text{Int}N(A))$ , where  $N(A)$  is the tubular neighborhood of  $A$  in  $S^4$  and  $\text{Int}N(A)$  the interior of  $N(A)$ . Let  $\mathbb{N}_{\geq 2} = \mathbb{N} - \{1\}$ . Let  $k \in \mathbb{N}_{\geq 2}$ . Let  $G$  be a group. Let  $G_k = [G, G_{k-1}]$  and  $G_1 = G$ . If  $G = \pi_L$ , let  $\pi_{L,k}$  denote  $G_k$  (we use ‘,’ in order to avoid confusion). Let  $L$  and  $L'$  be cobordant surface-links. It holds that for any  $k \in \mathbb{N}$ ,  $\pi_L/\pi_{L,k} \cong \pi_{L'}/\pi_{L',k}$ . In this paper we say that  $\pi_L$  and  $\pi_{L'}$  are *Stallings-equivalent* if for any  $k \in \mathbb{N}$ ,  $\pi_L/\pi_{L,k} \cong \pi_{L'}/\pi_{L',k}$ .

In §7 we submit a new way to detect surface-link-cobordism. We introduce terminologies, ‘ $n$ -covering-link ( $n \in \mathbb{N}$ )’, ‘alinking-equivalent’, and ‘weakly alinking-equivalent’. We prove the following: Suppose that surface-links  $\mathcal{L}$  and  $\mathcal{L}'$  with a condition are cobordant and that  $p$  is a prime power. Then ‘the  $p$ -covering-link of  $\mathcal{L}$ ’ and that of  $\mathcal{L}'$  under a condition are ‘weakly alinking-equivalent’. See §7 for the precise statement. This result is a theme of this paper.

We state our main result. There are two surface-links with the following properties. We cannot detect whether  $\mathcal{L}$  and  $\mathcal{L}'$  are non-cobordant by using Orr invariant, Cochran sequence, Sato-Levine invariant, the alinking number, and Stallings-equivalence which is defined as above. However our new way can detect it. See Main Theorem 1.4 for detail.

**Main Theorem 1.4.** *There are  $(S^2, S^2, T^2)$ -links,  $\mathcal{L} = (P, Q, R)$  and  $\mathcal{L}' = (P', Q', R')$ , with the following properties. Call all 2-component-sublinks of  $\mathcal{L}$  (resp.  $\mathcal{L}'$ ),  $\mathcal{L}_1 = (Q, R)$ ,  $\mathcal{L}_2 = (P, R)$ ,  $\mathcal{L}_3 = (P, Q)$ ,  $\mathcal{L}_{-1} = (R, Q)$ ,  $\mathcal{L}_{-2} = (R, P)$ , and  $\mathcal{L}_{-3} = (Q, P)$  (resp.  $\mathcal{L}'_1 = (Q', R')$ ,  $\mathcal{L}'_2 = (P', R')$ ,  $\mathcal{L}'_3 = (P', Q')$ ,  $\mathcal{L}'_{-1} = (R', Q')$ ,  $\mathcal{L}'_{-2} = (R', P')$ , and  $\mathcal{L}'_{-3} = (Q', P')$ .) Let  $\mathcal{S} = \{1, 2, 3, -1, -2, -3\}$ .*

- (1) *Let  $* \in \mathcal{S}$ . The alinking number of  $\mathcal{L}_*$  is the same as that of  $\mathcal{L}'_*$ .*
- (2) (i) *The 3-component-Sato-Levine invariant of  $\mathcal{L}$  is the same as that of  $\mathcal{L}'$ .*  
(ii) *Let  $* \in \mathcal{S}$ . The 2-component-Sato-Levine invariant of  $\mathcal{L}_*$  is the same as that of  $\mathcal{L}'_*$ .*
- (3) *Let  $* \in \mathcal{S}$ . The Cochran sequence of  $\mathcal{L}_*$  is the same as that of  $\mathcal{L}'_*$ .*
- (4) (i)  *$\pi_{\mathcal{L}}$  and  $\pi_{\mathcal{L}'}$  are Stallings-equivalent.*  
(ii) *Let  $* \in \mathcal{S}$ .  $\pi_{\mathcal{L}_*}$  and  $\pi_{\mathcal{L}'_*}$  are Stallings-equivalent.*
- (5) (i) *Let  $\# \in \mathbb{N}_{\geq 2} \cup \{\omega\}$ . The following two are equivalent.*  
(I) *We can define the Orr invariant  $\theta_{\#}(\mathcal{L}, \tau)$  for a meridian  $\tau$  and for the element  $\#$ .*  
(II) *We can define  $\theta_{\#}(\mathcal{L}', \tau')$  for a meridian  $\tau'$  and for the element  $\#$ .*  
*Suppose that (I) (resp. (II)) holds. Then we have  $\theta_{\#}(\mathcal{L}) = \theta_{\#}(\mathcal{L}') = 0$ .*  
(ii) *Let  $* \in \mathcal{S}$ . We can define the Orr invariant  $\theta_{\#}(\mathcal{L}_*, \alpha)$  (resp.  $\theta_{\#}(\mathcal{L}'_*, \alpha')$ ) for all meridians  $\alpha$  (resp.  $\alpha'$ ) and for all elements  $\# \in \mathbb{N}_{\geq 2} \cup \{\omega\}$ , and we have  $\theta_{\#}(\mathcal{L}_*) = \theta_{\#}(\mathcal{L}'_*) = 0$ .*
- (6) *Our new way in §7 implies that  $\mathcal{L}$  and  $\mathcal{L}'$  are non-cobordant.*

**Note.** We will define many links in this paper and we need many letters to represent them. So we use not only Roman-type letters but also calligraphic letters for them.

## 2. THE ALINKING NUMBER

The alinking number was introduced in [16]. Let  $l \in \mathbb{N}_{\geq 2}$ . Let  $L = (K_1, \dots, K_l)$  be a surface- $(F_1, \dots, F_l)$ -link in  $S^4$ . For any  $i \in \{1, \dots, l\}$ , take any circle embedded in  $K_i$ . Give any orientation to the circle. For any distinct  $i, j \in \{1, \dots, l\}$ , consider the linking number of the circle and  $K_j$  in  $S^4$ . Make a set of all of the linking number. Then the set is regarded as  $n \cdot \mathbb{Z}$  for a number  $n \in \{0\} \cup \mathbb{N}$ . Note that if  $n = 0$  then the set is  $\{0\}$ . We call this number  $n$  the *alinking number*  $\text{alk}(K_i \subset L, K_j \subset L)$  of  $K_i$  in  $L$  around  $K_j$  in  $L$ . Note that  $\text{alk}(K_1 \subset L, K_2 \subset L)$  is not equal to  $\text{alk}(K_2 \subset L, K_1 \subset L)$  in general.

Let  $L = (K_1, K_2)$  be a surface- $(S^2, F)$ -link in  $S^4$ . Then  $\text{alk}(K_1 \subset L, K_2 \subset L) = 0$ . Thus we let the alinking number of  $L$  be  $\text{alk}(K_2 \subset L, K_1 \subset L)$ .

A surface- $(F_1, \dots, F_\mu)$ -link  $L = (K_1, \dots, K_\mu)$  is called a *semi-boundary link* if for each  $i$ ,  $[K_i] = 0 \in H_2(S^4 - \amalg_{j \neq i, 1 \leq j \leq \mu} K_j; \mathbb{Z})$ . For any distinct  $i, j$ , a Seifert hypersurface  $V_i$  for  $K_i$  is called a *special Seifert hypersurface* if  $V_i \cap K_j = \emptyset$ . It is well-known that a surface- $(F_1, \dots, F_\mu)$ -link  $L = (K_1, \dots, K_\mu)$  is a semi-boundary link if and only if there is a special Seifert hypersurface  $V_i$  for  $K_i$  for each  $i$ .

**Proposition 2.1.** ([16, §2].) *Let  $L = (K_1, K_2)$  be a surface- $(F_1, F_2)$ -link. Then (1) – (3) are equivalent each other.*

- (1)  $L$  is semi-boundary.
- (2)  $L$  is cobordant to a semi-boundary link.
- (3) Both  $\text{alk}(K_1 \subset L, K_2 \subset L)$  and  $\text{alk}(K_2 \subset L, K_1 \subset L)$  are zero.

**Corollary 2.2.** ([16, §2].) *If a surface- $(F_1, F_2)$ -link  $L = (K_1, K_2)$  is cobordant to the standard link,  $L$  is semi-boundary.*

Proposition 2.1 implies that if two surface-links are cobordant and the alinking number of one of the two is zero, then that of the other is zero. In [13, Proposition 7.10] the author generalized it and proved the following. The proof is short so it is cited here.

**Theorem 2.3.** *If surface- $(F_a, F_b)$ -links,  $L_0 = (K_{0a}, K_{0b})$  and  $L_1 = (K_{1a}, K_{1b})$ , are cobordant, then*

$$\text{alk}(K_{0a} \subset L_0, K_{0b} \subset L_0) = \text{alk}(K_{1a} \subset L_1, K_{1b} \subset L_1)$$

and

$$\text{alk}(K_{0b} \subset L_0, K_{0a} \subset L_0) = \text{alk}(K_{1b} \subset L_1, K_{1a} \subset L_1).$$

**Proof of Theorem 2.3.** Take  $S^4 \times [0, 1]$ . Let  $j \in \{0, 1\}$ . Regard  $L_j$  as a submanifold of  $S^4 \times \{j\}$ . Take an embedding map  $f : (F_a \times [0, 1]) \amalg (F_b \times [0, 1]) \hookrightarrow S^4 \times [0, 1]$  which gives cobordism between  $L_0$  and  $L_1$ . Let  $V_{jb}$  be a Seifert hypersurface for  $K_{jb} \subset S^4 \times \{j\}$ , where  $K_{ja} \cap V_{jb} \neq \emptyset$  may hold. Note that there is an embedded compact oriented 4-manifold  $W_b \subset S^4 \times [0, 1]$  whose boundary is  $V_{0b} \cup f(F_b \times [0, 1]) \cup V_{1b}$ . Take any embedded circle  $C \subset F_a$  and any  $t \in [0, 1]$ . By using  $f(C \times \{t\}) \cap W_b$ , we can prove that  $\text{alk}(K_{0a} \subset L_0, K_{0b} \subset L_0) = \text{alk}(K_{1a} \subset L_1, K_{1b} \subset L_1)$  holds. By replacing  $a$  with  $b$  in the above proof, we can prove that  $\text{alk}(K_{0b} \subset L_0, K_{0a} \subset L_0) = \text{alk}(K_{1b} \subset L_1, K_{1a} \subset L_1)$  holds.  $\square$

### 3. THE SATO-LEVINE INVARIANT

The Sato-Levine invariant was defined by Sato ([16]) and by Levine (unpublished). Let  $L = (K_1, K_2)$  be a semi-boundary surface- $(F_1, F_2)$ -link. For each  $i \in \{1, 2\}$ , let  $V_i$  be a special Seifert hypersurface for  $K_i$ . We can suppose that  $V_1$  intersects  $V_2$  transversely. Note that  $G = V_1 \cap V_2$  is a (not necessarily connected) closed oriented surface. By using  $G$ ,  $V_1$  and  $V_2$ , we can define Thom-Pontrjagin map  $p : S^4 \rightarrow S^2$  such that the inverse image of

a regular value is  $G$ . The (*2-component*) *Sato-Levine invariant*  $\beta(L)$  is  $[p] \in \pi_4(S^2) \cong \mathbb{Z}_2$ . In [10] the author proved that we can regard  $\beta(L)$  as an element  $\in \Omega_2^{\text{spin}} \cong \mathbb{Z}_2$ , where  $\Omega_2^{\text{spin}}$  is the second spin cobordism group. We can define the *3-component Sato-Levine invariant*  $\beta(L) \in \pi_4(S^3) \cong \mathbb{Z}_2$  for any semi-boundary surface- $(F_1, F_2, F_3)$ -link  $L = (K_1, K_2, K_3)$ . We can define the *4-component Sato-Levine invariant*  $\beta(L) \in \pi_4(S^4) \cong \mathbb{Z}$  for any semi-boundary surface- $(F_1, F_2, F_3, F_4)$ -link  $L = (K_1, K_2, K_3, K_4)$ . If  $m \geq 5$ , it is nonsense to define the Sato-Levine invariant for  $m$ -component surface-link  $L$ .

[15, 16] proved that there is a  $(S^2, T^2)$ -link whose Sato-Levine invariant is nontrivial.

In [14] Orr proved outstanding results: Let  $m \in \{2, 3, 4\}$ . If  $L$  is an  $m$ -component spherical 2-link, the Sato-Levine invariant of  $L$  is zero ([14, Theorem 4.1]). If  $L$  is a 4-component (not necessarily spherical) surface-link, the Sato-Levine invariant of  $L$  is zero ([14, Theorem 3.5]).

#### 4. THE COCHRAN SEQUENCE

The Cochran sequence was defined in [4]. Let  $* \in \mathbb{N}$ . Let  $F$  (resp.  $G, F_*$ ) be a connected closed oriented surface through this section.

**Definition.** A  $(S^2, F)$ -link  $L = (K_1, K_2) \subset S^4$  and a  $(S^2, G)$ -link  $L' = (K'_1, K'_2) \subset S^4$  are *weak-cobordant* if they satisfy the following conditions:

(1) The alinking number of  $L$  (resp.  $L'$ ) is zero. Hence there are Seifert hypersurfaces  $Z$  (resp.  $Z'$ ) for  $K_1$  (resp.  $K'_1$ ) in  $S^4 - \text{Int}N(K_2)$  (resp.  $S^4 - \text{Int}N(K'_2)$ ), where  $N(K_*)$  is the tubular neighborhood of  $K_* \subset S^4$ .

(2) Let  $I = [0, 1]$ . There is a compact oriented 3-dimensional submanifold  $W \amalg Y \subset S^4 \times I$  such that

$$(a) \partial(W \amalg Y) = L \amalg L'$$

$$(b) W = S^2 \times I$$

(c) The closed compact oriented 3-dimensional submanifold  $Z \cup W \cup Z'$  bounds a compact oriented 4-dimensional submanifold  $Q \subset \overline{(S^4 \times I) - N(Y)}$ , where  $N(Y)$  is the tubular neighborhood of  $Y \subset S^4 \times I$ .  
 $Q \cap (S^4 - \text{Int}N(K_2)) = Z$ .  $Q \cap (S^4 - \text{Int}N(K'_2)) = Z'$ .

It is trivial that the following holds.

**Proposition 4.1.** *If  $(S^2, T^2)$ -links,  $L = (K_1, K_2)$  and  $L' = (K'_1, K'_2) \subset S^4$  are cobordant and if the alinking number of  $L$  is zero, then they are weak-cobordant.*

**Definition.** Let  $L = (K_1, K_2)$  be a  $(S^2, T^2)$ -link whose alinking number is zero. By [4, Theorem 4.1] there is a Seifert hypersurface  $V_i$  for  $K_i$  ( $i = 1, 2$ ) such that  $V_1$  intersects  $V_2$  transversely and such that  $V_1 \cap V_2 = J$  is a closed oriented connected surface. The *derivative link*  $D(L)$  of  $L$  is the weak-cobordism class of the surface-link  $(K_1, J)$ .

**Theorem.** ([4, Theorem 4.2].) *If a  $(S^2, F)$ -link  $L = (K_1, K_2) \subset S^4$  and a  $(S^2, G)$ -link  $L' = (K'_1, K'_2) \subset S^4$  are weak-cobordant, then  $D(L)$  and  $D(L')$  are weak-cobordant.*

**Theorem.** ([4, Corollary 4.3].) *For any  $(S^2, F)$ -link  $L = (K_1, K_2)$  whose alinking number is zero, the sequence of links*

$$D^0(L) = L, D(L), D^2(L) = D(D(L)), D^3(L) = D(D(D(L))) \dots$$

*is well defined in the category of links modulo weak-cobordant.*

**Definition.** Let  $L = (K_1, K_2)$  be a  $(S^2, F)$ -link whose alinking number is zero. Note that for any natural number  $n$ ,  $D^n(L)$  is a  $(S^2, F_n)$ -link whose alinking number is zero. Note that we can define the Sato-Levine invariant  $\beta(D^n(L)) (n \in \mathbb{N})$ . Let  $\beta^{n+1}(L) = \beta(D^n(L)) (n \in \mathbb{N})$  and  $\beta^1(L) = \beta(L)$ . The *Cochran sequence* for  $L$  is the sequence  $\{\beta^n(L)\}_{n \in \mathbb{N}}$ .

**Theorem.** ([4, Corollary 5.2].) *Let  $L = (K_1, K_2)$  be a  $(S^2, F)$ -link whose alinking number is zero. The Cochran sequence  $\{\beta^n(L)\}_{n \in \mathbb{N}}$  is weak-cobordism invariant.*

It is trivial that the following holds.

**Theorem 4.2.** *Suppose that  $(S^2, T^2)$ -links,  $L$  and  $L'$ , are cobordant and that the alinking number of  $L$  is zero. Then the Cochran sequences,  $\{\beta^n(L)\}_{n \in \mathbb{N}}$  and  $\{\beta^n(L')\}_{n \in \mathbb{N}}$ , are equivalent.*

[4] proved that there is a  $(S^2, T^2)$ -link with a nontrivial Cochran sequence.

## 5. THE ORR INVARIANT

In [14, §2] Orr defined the Orr invariant  $\theta_k$  for any  $k \in \mathbb{N}_{\geq 2}$  and  $\theta_\omega$  in Definition 5.1 for codimension two closed oriented submanifolds  $L \subset S^{n+2}$  if  $L$  is a disjoint union of connected homology spheres. He used his invariant and proved [14, Theorem 4.1], which is explained in §3. Furthermore, in [14, the first few lines of §5], he stated that  $\theta_k$  and  $\theta_\omega$  can be defined in some other cases which include ones in Definition 5.1.

**Definition 5.1.** Let  $m \in \mathbb{N}_{\geq 2}$ . Let  $L = (K_1, \dots, K_m)$  be an  $m$ -component surface-link. Let  $E_L = S^4 - \text{Int}N(L)$ . Note that  $N(L) = L \times D^2$ . Note that  $\partial E_L = L \times S^1 = \partial N(L)$ .

Let  $p$  be a base point of  $S^4$ . Let  $S_1^1 \vee \dots \vee S_m^1$  be a bouquet. Let  $b = S_1^1 \cap \dots \cap S_m^1$  be a base point of  $S^1 \vee \dots \vee S^1$ . Let  $(L, \tau)$  be an  $m$ -component surface-link with a fixed meridian. The meridian of  $L$  defines a continuous map

$$\tau : (S_1^1 \vee \dots \vee S_m^1, b) \rightarrow (E_L, p).$$

Here, we use  $\tau$  again. Let  $i \in \{1, \dots, m\}$ . Note that  $\tau|_{S_i^1}$  defines the meridian of  $K_i$ .

Let  $\pi_L$  denote  $\pi_1 E_L$ . Let  $F = \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_m$ . Suppose that the condition (1) (resp. (2))

holds.

(1)  $\pi_L/\pi_{L,k} = F/F_k$  for a natural number  $k$ .

(2)  $\lim_{\infty \leftarrow k} \pi_L/\pi_{L,k} = \lim_{\infty \leftarrow k} F/F_k$

(By [2, Hopf's theorem] and [17], we have the following: if  $L$  is a disjoint union of connected homology spheres, the congruence in (1) holds for all natural numbers  $k$  and the condition (2) holds.)

Let  $K(G, 1)$  be the Eilenberg-MacLane space for any group  $G$ . Let  $\bar{\tau}$  denote the induced homotopy-type-equivalence map  $K(F/F_k, 1) \rightarrow K(\pi_L/\pi_{L,k}, 1)$  (resp.  $K(F/F_\omega, 1) \rightarrow K(\pi_L/\pi_{L,\omega}, 1)$ ). Let  $\phi_k : E_L \rightarrow K(F/F_k, 1)$  (resp.  $\phi_\omega : E_L \rightarrow K(\lim_{\infty \leftarrow k} F/F_k, 1)$ ) be the composition of 'the map  $E_L \rightarrow K(\pi_L/\pi_{L,k}, 1)$  (resp.  $E_L \rightarrow K(\pi_L/\pi_{L,\omega}, 1)$ ) realizing  $\pi_L \rightarrow \pi_L/\pi_{L,k}$  (resp.  $\pi_L \rightarrow \pi_L/\pi_{L,\omega}$ )' with the map  $(\bar{\tau})^{-1}$ . Thus we define the following homomorphism naturally:

$$\pi_1 K_i \rightarrow \pi_1 K_i \times \pi_1 S^1 \cong \pi_1(K_i \times S^1) \rightarrow \pi_L \rightarrow \pi_L/\pi_{L,k} \rightarrow F/F_k$$

$$(\text{resp. } \pi_1 K_i \rightarrow \pi_1 K_i \times \pi_1 S^1 \cong \pi_1(K_i \times S^1) \rightarrow \pi_L \rightarrow \lim_{\infty \leftarrow k} \pi_L/\pi_{L,k} \rightarrow \lim_{\infty \leftarrow k} F/F_k)$$

This induces a homomorphism  $\pi_1 K_i \rightarrow F/F_k$  (resp.  $\pi_1 K_i \rightarrow \lim_{\infty \leftarrow k} F/F_k$ ). Suppose that we have the condition

(3)  $\pi_1 K_i \rightarrow F/F_k$  (resp.  $\pi_1 K_i \rightarrow \lim_{\infty \leftarrow k} F/F_k$ ) is the zero map.

(If  $L$  is a disjoint union of connected homology spheres, the condition (3) holds. See [14, the part from the last line of page 546 to the first line of page 547]. )

The quotient homomorphism

$$\psi_k : F \rightarrow F/F_k$$

induces an inclusion of Eilenberg-MacLane spaces

$$\psi_k : S_1^1 \vee \dots \vee S_m^1 = K(F, 1) \rightarrow K(F/F_k, 1).$$

Let  $K_k$  be the mapping cone of  $\psi_k$ . Note that  $K_k$  is simply-connected.

The homomorphisms  $\psi_k$  induce a homomorphism

$$\psi_\omega : F \rightarrow \lim_{\infty \leftarrow k} F/F_k$$

and an inclusion of the spaces

$$\psi_\omega : S_1^1 \vee \dots \vee S_m^1 = K(F, 1) \rightarrow K(\lim_{\infty \leftarrow k} F/F_k, 1).$$

Let  $K_\omega$  be the mapping cone of  $\psi_\omega$ .

The condition (3) and the property of the Eilenberg-MacLane space imply the following commutative diagram.



$$\begin{array}{ccc}
K_i \times S^1 & \xrightarrow{\phi_k|_{K_i \times S^1}} & K(F/F_k, 1) \\
\downarrow \text{projection} & \circlearrowleft & \\
S^1 & \nearrow & 
\end{array}$$

Therefore  $\phi_k$  (resp.  $\phi_\omega$ ) extends canonically to a continuous map

$$\begin{aligned}
& \bar{\phi}_k : S^4 \rightarrow K_k \\
& (\text{resp. } \bar{\phi}_\omega : S^4 \rightarrow K_\omega).
\end{aligned}$$

Define the *Orr invariant*  $\theta_k(L, \tau)$  (resp.  $\theta_\omega(L, \tau)$ ) to be  $[\bar{\phi}_k] \in \pi^4(K_k)$  (resp.  $[\bar{\phi}_\omega] \in \pi^4(K_\omega)$ ), where  $k \in \mathbb{N}_{\geq 2}$ .

If  $\theta_*$  is defined for  $*$   $\in \Lambda$  which is an infinite set, the sequence  $\{\theta_*\}_{* \in \Lambda}$  is called *Orr sequence*.

[14, Theorem in §5] claimed that there is a surface-link with a nontrivial Orr invariant. We have the following theorem.

**Theorem 5.2.** ([14, Theorem 2.1.(iv)].) *Let  $k \in \mathbb{N}_{\geq 2} \cup \{\omega\}$ . We have the following:  $\theta_k(L, \tau) = 0$  for a choice  $\tau$  of meridians for  $L$  if and only if  $\theta_k(L, \tau') = 0$  for any other choice  $\tau'$  of meridians for  $L$ .*

By Theorem 5.2, the notation, ‘ $\theta_k(L) = 0$ ’, makes sense. (Note that we omit a meridian in ( ) of  $\theta_k$ ( ).)

## 6. BOUNDARY LINKS AND SPUN SURFACE-LINKS

We use boundary links and spun surface-links in order to make examples in the proof. We defined boundary links in §1. We now define spun surface-links.

**Definition 6.1.** We say that a surface-link  $L$  is a *spun surface-link* if  $L$  is (not necessarily order-preserving) equivalent to the surface link  $L'$  which is obtained as follows.

Regard  $S^4$  as  $\mathbb{R}^4 \cup \infty$ . Regard  $\mathbb{R}^4 = \{(x, y, z, w) | x, y, z, w \in \mathbb{R}\}$ . Regard  $\mathbb{R}^4 = \{(x, y, z, w) | x, y, z, w \in \mathbb{R}\}$  as the result of rotating  $Q$   
 $= \{(x, y, z, w) | x, y \in \mathbb{R}, z \geq 0, w = 0\}$  around  $A = \{(x, y, z, w) | x, y \in \mathbb{R}, z = 0, w = 0\}$ .

Let  $I_1, \dots, I_p$  be the intervals. Let  $f : I_1 \amalg \dots \amalg I_p \amalg S_1^1 \amalg \dots \amalg S_q^1 \hookrightarrow Q$  be an embedding map such that for each  $*$ ,  $f(I_*) \cap A$  is the two points  $f(\partial I_*)$ . Rotate  $\text{Im} f$  around  $A$  together when we rotate  $F$  around  $A$  and obtain  $\mathbb{R}^4$ . Thus we obtain a surface-link  $L' \subset S^4$ .

We prove some properties of boundary links and spun surface-links.

**Proposition 6.2.** *Let  $L = (K_1, \dots, K_n)$  be a boundary surface- $(F_1, \dots, F_n)$ -link. Then  $L = (K_1, \dots, K_n)$  is cobordant to the standard link.*

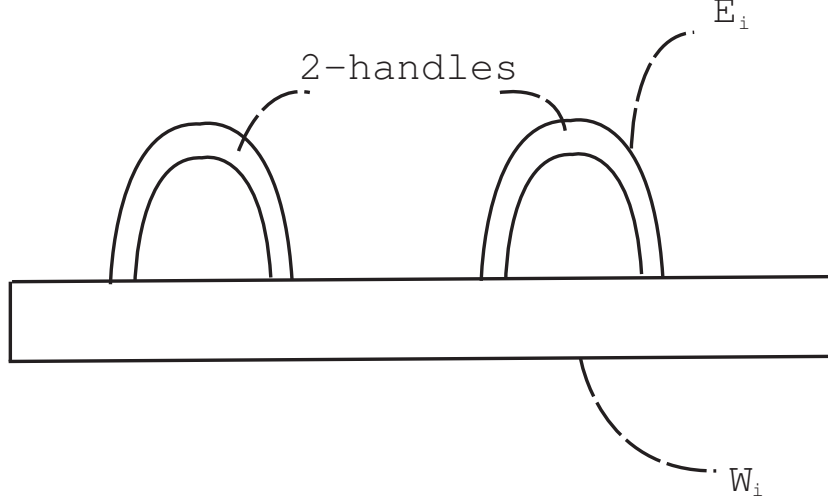


FIGURE 6.1. An example of the handle decomposition of  $O_i$

**Proof of Proposition 6.2.** Let  $L' = (K'_1, \dots, K'_n)$  be the standard surface- $(F_1, \dots, F_n)$ -link. Embed a 4-ball  $B$  (resp.  $B'$ ) in  $S^4$  such that  $B \cap B' = \emptyset$ . Take  $L$  (resp.  $L'$ ) in  $B$  (resp.  $B'$ ). Let  $V_1 \amalg \dots \amalg V_n \subset B$  (resp.  $V'_1 \amalg \dots \amalg V'_n \subset B'$ ) be a disjoint embedded 3-manifold such that  $V_i \cap V_j = \emptyset$  (resp.  $V'_i \cap V'_j = \emptyset$ ) for all distinct  $i, j$ , and such that  $V_i$  (resp.  $V'_i$ ) is a Seifert hypersurface for  $K_i$  (resp.  $K'_i$ ) for each  $i$ . Suppose that  $V'_i$  is an oriented genus  $g_i$  handlebody for each  $i$ , where  $g_i$  is the genus of  $F_i$ . Connect  $V_i$  and  $V'_i$  by an embedded 3-dimensional 1-handle  $h_i^1 \subset S^4$  for each  $i$  such that  $h_i^1 \cap h_j^1 = \emptyset$  for all distinct  $i, j$ , and call the result  $W_i$ . See [1, 8, 18] for handles, surgery and their associated terms. Note that the boundary of  $W_1 \amalg \dots \amalg W_n$  is a surface- $(F_1 \sharp F_1, \dots, F_n \sharp F_n)$ -link, where  $\sharp$  denotes the connected-sum.

Since  $\Omega_3^{\text{Spin}} \cong 0$  (see [8]), there is a compact oriented spin 4-manifold  $O_i$  with a handle decomposition (see Figure 6.1)

$$O_i = (W_i \times [0, 1]) \cup (4\text{-dimensional 2-handles}) \cup (E_i \times [0, 1])$$

to satisfy the following conditions:  $W_i$  is the bottom and  $E_i$  is the top.  $\partial O_i = (W_i \times \{0\}) \cup (E_i \times \{1\})$ .  $E_i = (F_i - (\text{an embedded open 2-disc})) \times [0, 1]$ .

Note that, here, we make  $O_i$  as an absolute manifold not as a submanifold. We make a submanifold which is diffeomorphic to  $O_i$  from now.

Recall that  $L$  and  $L'$  are in  $S^4$ . Take  $S^4 \times [0, 1]$ . Take  $(S^4, L \amalg L') \times [0, 1]$ , where  $(X, Y)$  denotes a pair of a manifold  $X$  and a submanifold  $Y \subset X$ . Take the tubular neighborhood  $N(W_i) = W_i \times [-1, 1]$  of  $W_i$  in  $S^4$ . Take  $(S^4, (W_1 \times [-1, 1]) \amalg \dots \amalg (W_n \times [-1, 1])) \times [0, 1]$ . Take  $O_i \times [-1, 1]$ . Make these  $S^4 \times [0, 1]$  and  $O_i \times [-1, 1]$  into a 5-manifold as follows:

Identify  $W_i \times [-1, 1] \times [0, 1] \subset S^4 \times [0, 1]$  with  $W_i \times [0, 1] \times [-1, 1] \subset O_i \times [-1, 1]$  so that for each  $s \in [-1, 1]$  and each  $t \in [0, 1]$ ,  $W_i \times \{s\} \times \{t\} \subset S^4 \times [0, 1]$  coincides with  $W_i \times \{t\} \times \{s\} \subset O_i \times [-1, 1]$ . Thus the resultant 5-manifold  $M$  has a handle decomposition

$$M = (S^4 \times [0, 1]) \cup (\text{5-dimensional 2-handles}).$$

Since  $M$  is a spin oriented 5-manifold,

$$M \cong (\natural^\mu S^2 \times B^3) - (\text{an open 5-ball}),$$

where  $\natural$  denotes the boundary-connected-sum and  $\mu$  is a natural number.

Note that  $M$  can be embedded in  $B^5$  so that  $S^4 \times \{0\}$  coincides with  $\partial B^5$ . Recall that  $L \subset B \subset S^4 = \partial B^5 \subset B^5$  and that  $L' \subset B' \subset S^4 = \partial B^5 \subset B^5$ . Recall that  $E_i \cup h_i^1 \subset B^5$  and that  $E_i \cup h_i^1$  is diffeomorphic to  $F_i \times [-1, 1]$ . Therefore we can embed  $F_i \times [-1, 1] \subset B^5$  such that  $F_i \times \{-1\}$  (resp.  $F_i \times \{1\}$ ) coincides with  $K_i$  of  $L \subset B \subset S^4$  (resp.  $K'_i$  of  $L' \subset B' \subset S^4$ ) for each  $i$  and such that  $(F_i \times [-1, 1]) \cap (F_j \times [-1, 1]) = \emptyset$  for all distinct  $i, j$ . Hence Proposition 6.2 holds.  $\square$

By Proposition 6.2, Theorem 6.3.(1)  $\Rightarrow$  Theorem 6.3.(2) is trivial. However the converse is not true in general.

**Theorem 6.3.** *Let  $L = (K_1, \dots, K_n) \subset S^4$  be a surface- $(F_1, \dots, F_n)$ -link. Then (1)  $\Rightarrow$  (2) is true, but (2)  $\Rightarrow$  (1) is false in general.*

(1)  $L$  is cobordant to a boundary-link.

(2) Suppose that  $L \subset S^4 = \partial B^5 \subset B^5$ . Let  $g_i$  be the genus of  $F_i$ . There is an embedded submanifold  $O_1 \amalg \dots \amalg O_n \subset B^5$  with the following properties. Let  $i, j \in \{1, \dots, n\}$ .

(i)  $O_i$  is an oriented genus  $g_i$  handlebody.

(ii)  $O_i \cap O_j = \emptyset$  for all distinct  $i, j$ .

(iii)  $O_1 \amalg \dots \amalg O_n$  meets  $\partial B^5$  transversely.  $O_i \cap \partial B^5 = \partial O_i$ .

$\partial O_1 \amalg \dots \amalg \partial O_n$  (resp.  $\partial O_i$ ) in  $\partial B^5$  is  $L$  (resp.  $K_i$ ).

**Note.** It is trivial that if  $g_i = 0$  for each  $i$ , (2)  $\Rightarrow$  (1) is true.

**Proof of Theorem 6.3.** We prove that Theorem 6.3.(2)  $\Rightarrow$  Theorem 6.3.(1) is false in general. We define a  $(S^2, T^2)$ -link  $L = (K, J)$  as follows: Let  $K$  be the trivial spherical 2-knot  $\subset S^4$ . Let  $N(K)$  be the tubular neighborhood of  $K$  in  $S^4$ . Note that  $N(K) = K \times D^2$ . Let  $C$  be an embedded circle  $\{*\} \times \partial D^2 \subset S^4$ . Embed a solid torus  $S^1 \times D^2$  so that  $S^1 \times \{\#\}$  coincides with  $C$ . Let  $J$  be an embedded torus  $\partial(S^1 \times D^2)$ . By the construction,  $L$  satisfies Theorem 6.3.(2). By the construction,  $\text{alk}(J, K) = 1$ . By Proposition 2.1 and Theorem 2.3,  $L$  does not satisfy Theorem 6.3.(1).  $\square$

**Note 6.4.** (1) Take a spun  $(S^2, T^2)$ -link  $P$  of Figure 6.2. There, we omit the  $x$ -,  $y$ -axes and draw the  $z$ -axis. We claim that  $P$  is equivalent to  $L$  in Proof of Theorem 6.3. *Reason.*

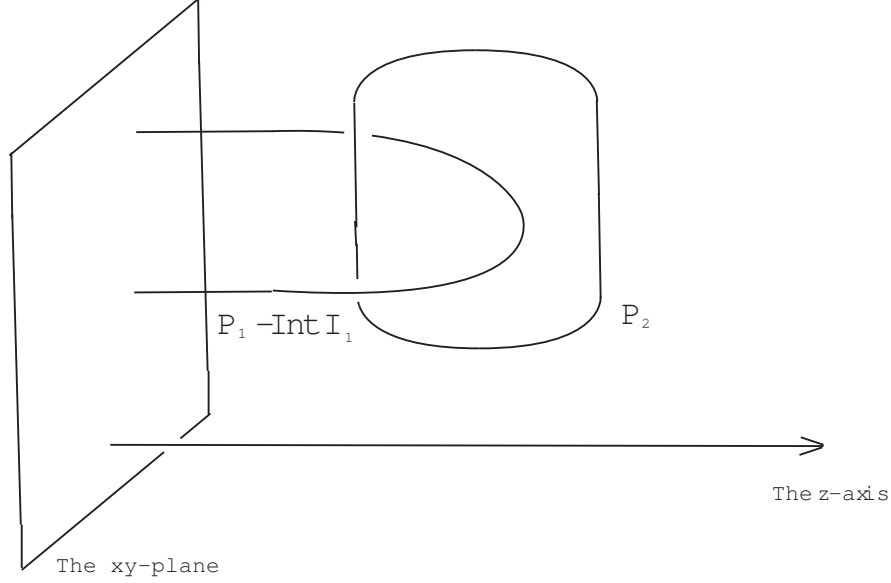


FIGURE 6.2. A 1-link  $(P_1, P_2)$  whose spun surface-link is a  $(S^2, T^2)$ -link with a nontrivial alinking-number.

Let  $(S^4$  in Theorem 6.3)–(a point  $\{\infty\}$ ) be  $\mathbb{R}^4$  in Proof of Theorem 6.3. Move  $C$  in Proof of Theorem 6.3 in  $S^4$  by an isotopy, and let  $C$  include  $\{\infty\}$ . Then  $L$  becomes  $P$ .

(2) Not all spun  $(S^2, T^2)$ -links are cobordant to the standard link because of the existence of  $P$  in Note 6.4.(1). On the contrary it holds that all spun  $m$ -component spherical 2-links are slice, that is, cobordant to the trivial link ( $m \in \mathbb{N}$ ). Furthermore we will prove in §9 that not all semi-boundary spun  $(S^2, T^2)$ -links are cobordant to the standard link.

**Theorem 6.5.** *Let  $L$  be a spun surface-link. If for a meridian  $\tau$  and for an element  $k \in \mathbb{N}_{\geq 2} \cup \{\omega\}$ , we can define the Orr invariant  $\theta_k(L, \tau)$ , then we have  $\theta_k(L) = 0$ .*

**Lemma 6.6.** *Let  $L$  be a spun surface-link  $\subset S^4$ . Then there is an embedding map  $g : D^3 \amalg \dots \amalg D^3 \amalg (D^2 \times S^1) \amalg \dots \amalg (D^2 \times S^1) \hookrightarrow B^5$  with the following properties:*

(1)  *$g$  meets  $\partial B^5$  transversely. Call  $D^3 \amalg \dots \amalg D^3 \amalg (D^2 \times S^1) \amalg \dots \amalg (D^2 \times S^1)$ ,  $Y$ .  $g^{-1}(\partial B^5)$  is  $\partial Y$ .  $g(\partial Y)$  in  $\partial B^5$  is a surface-link which is (not necessarily order-preserving) equivalent to  $L$ .*

(2) *Call the submanifold  $g(Y) \subset B^5$ ,  $M$ . Let  $E_M = \overline{B^5 - N(M)}$ , where  $N(M)$  is the tubular neighborhood of  $M$  in  $B^5$  and  $\overline{abc}$  denotes the closure in  $B^5$ . Let  $\pi_M$  denote  $\pi_1(E_M)$ . Recall that  $E_L = S^4 - \text{Int}N(L)$  and  $\pi_L$  which are defined in §5. The inclusion map  $E_L \hookrightarrow E_M$  induces an isomorphism map  $\pi_L \cong \pi_M$ .*

**Proof of Lemma 6.6.** Take an embedding map  $f : I_1 \amalg \dots \amalg I_p \amalg S_1^1 \amalg \dots \amalg S_q^1 \hookrightarrow Q = \{(x, y, z, w) | x, y \in \mathbb{R}, z \geq 0, w = 0\}$ , which makes a surface-link which is (not necessarily order-preserving) equivalent to  $L$ , as in Definition 6.1 for appropriate natural numbers  $p$  and  $q$ . As we stated there, we regard  $\mathbb{R}^4 = \{(x, y, z, w) | x, y, z, w \in \mathbb{R}\}$  as the result of rotating  $Q = \{(x, y, z, w) | x, y \in \mathbb{R}, z \geq 0, w = 0\}$  around  $A = \{(x, y, z, w) | x, y \in \mathbb{R}, z = 0, w = 0\}$ .

Let  $\mathbb{R}_{z \geq 0}^4 = \{(x, y, z, w) | z \geq 0\}$ . Recall that  $L \subset \mathbb{R}^4$ . Let  $L_{z \geq 0} = L \cap \mathbb{R}_{z \geq 0}^4$ . Regard  $\mathbb{R}_{v \geq 0}^5 = \{(x, y, z, w, v) | v \geq 0\}$  as the result of rotating  $\mathbb{R}_{z \geq 0}^4$  through 180 degrees around  $\{(x, y, z, w) | z = 0\}$  as the axis. When we carry out this half rotation, we rotate  $L_{z \geq 0}$  together. Call the resultant submanifold contained in  $\mathbb{R}_{v \geq 0}^5$ ,  $M$ . Note that we have the following:  $M$  is a disjoint union of  $p$  copies of  $D^3$  and  $q$  copies of  $D^2 \times S^1$ .  $M \cap \{(x, y, z, w, v) | v = 0\} = \partial M$ .  $\partial M$  in  $\{(x, y, z, w, v) | v = 0\} \cup \{\infty\}$  is  $L$  if we give an appropriate order to its components.

By the construction,  $\overline{Q - N(\text{Im} f)}$  is homotopy type equivalent to  $\overline{\mathbb{R}_{z \geq 0}^4 - N(L_{z \geq 0})}$ , and to  $\overline{\mathbb{R}_{v \geq 0}^5 - N(M)}$ , and each homotopy-type-equivalence map is each natural inclusion. By using this homotopy type equivalence, Lemma 6.6 holds.  $\square$

**Proof of Theorem 6.5.** By the assumption, we can define the Orr invariant  $\theta_k(L, \tau)$  by using a continuous map  $\bar{\phi}_k : S^4 \rightarrow K_k$  as in §5. Recall that the map  $\bar{\phi}_k$  is made from  $\phi_k : E_L \rightarrow K(F/F_k, 1)$  (or  $\lim_{\infty \leftarrow k} K(F/F_k, 1)$ ).

Call  $f(\partial Y) \subset \partial B^5$ ,  $L$  again, and let  $L = (L_1, \dots, L_r)$ , where  $r = p + q$ . Let  $* \in \{1, \dots, r\}$ . Let  $M$  be a disjoint union of compact connected components  $M_1 \amalg \dots \amalg M_r$  such that  $\partial M_* = L_*$ . Consider the following commutative diagram of continuous maps. The maps are inclusion maps

$$\begin{array}{ccccc} M_i \times \partial D^2 & \rightarrow & M \times \partial D^2 & \rightarrow & E_M \\ \uparrow & & \uparrow & & \uparrow \\ \partial N(L_i) = L_i \times \partial D^2 & \rightarrow & \partial E_L & \rightarrow & E_L \end{array}$$

This induces the following commutative diagram of homomorphisms. The homomorphism represented by the left uparrow is called  $\xi$ .

$$\begin{array}{ccccc} \pi_1(M_i \times \partial D^2) & \rightarrow & \pi_1(M \times \partial D^2) & \rightarrow & \pi_M \\ \uparrow \xi & & \uparrow & & \uparrow \cong \\ \pi_1(L_i \times \partial D^2) & \rightarrow & \pi_1(\partial E_L) & \rightarrow & \pi_L \end{array}$$

By the assumption and Lemma 6.6, the homomorphism of the right uparrow  $\pi_M \rightarrow \pi_L$  is isomorphic. Therefore the map  $\phi_k$  extends to a map  $\Phi_k : E_M \rightarrow K(F/F_k, 1)$  (or  $\lim_{\infty \leftarrow k} K(F/F_k, 1)$ ).

We have  $\pi_1(L_i \times \partial D^2) = \pi_1(L_i) \times \pi_1(\partial D^2)$  and  $\pi_1(M_i \times \partial D^2) = \pi_1(M_i) \times \pi_1(\partial D^2)$ . It holds that  $\xi(\pi_1(L_i)) \subset \pi_1(M_i)$ . Furthermore this inclusion map  $\pi_1(L_i) \rightarrow \pi_1(M_i)$  is

onto. Therefore the condition (3) in §5 and the property of the Eilenberg-MacLane space imply the following commutative diagram.

$$\begin{array}{ccc}
M_i \times S^1 & \xrightarrow{\Phi_k|_{M_i \times S^1}} & K(F/F_k, 1) \\
\downarrow \text{projection} & \circlearrowleft & \\
S^1 & \nearrow & 
\end{array}$$

Therefore  $\Phi_k$  extends canonically to a continuous map  $\bar{\Phi}_k : B^5 \rightarrow K_k$  such that  $\bar{\Phi}_k|_{\partial B^5} = \bar{\phi}_k$ .

This completes the proof of Theorem 6.5.  $\square$

**Note.** The author could prove that if we replace ‘spun’ with ‘ribbon’ in Lemma 6.6, Lemma 6.6 holds. See Note 6.7.(2) and [13, the paragraph right before Corollary 4.14] for the definition of ribbon- $(S^2, T^2)$ -links.

**Note 6.7.** In [13] the author proved the following facts (1)-(3).

(1) Let  $A(t)$  represent the 1-st  $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of a  $(S^2, T^2)$ -link  $L$ . Then

$\left. \frac{A(t)}{(t-1)} \right|_{t=1} = 0$  if and only if the alinking number of  $L$  is zero.

(2) We say that a  $(S^2, T^2)$ -link  $L = (K_1, K_2)$  is *ribbon* if there is an immersion  $f : B \amalg H \looparrowright S^4$  with the following properties, where  $B$  is a 3-ball and  $H$  is an oriented genus one handlebody: The self-intersection of  $f$  consists of double points and is a disjoint union of 2-discs. Note that  $f^{-1}$ (each disc) is a disjoint union of two 2-discs. One of the two disc is included in the interior of  $B \amalg H$ . ‘(The other disc)  $\cap \partial(B \amalg H)$ ’ is  $\partial$ (the other disc).

Let  $A(t)$  represent the 1-st  $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of a ribbon- $(S^2, T^2)$ -link  $L$ . Then  $\left| \left. \frac{A(t)}{(t-1)} \right|_{t=1} \right|$  is the alinking number of  $L$ .

(3) Let  $A(t)$  and  $L$  be as in (2). Let  $B(t)$  represent the 1-st  $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of a ribbon- $(S^2, T^2)$ -link  $M$ . If the ribbon- $(S^2, T^2)$ -links  $L$  and  $M$  are cobordant, then

$$\left| \left. \frac{A(t)}{(t-1)} \right|_{t=1} \right| = \left| \left. \frac{B(t)}{(t-1)} \right|_{t=1} \right|.$$

The author does not know whether we can remove the condition ‘ribbon’ from (2) and (3).

## 7. OUR NEW WAY TO INVESTIGATE SURFACE-LINK-COBORDISM

In order to introduce our new way, we begin by defining terminologies.

**Definition 7.1.** Let  $n \in \mathbb{N}_{\geq 2}$ . Let  $\mathcal{K} = (K_1, \dots, K_n)$  and  $\mathcal{K}' = (K'_1, \dots, K'_n)$  be a surface-link of the standard 4-sphere. Let  $\{i, j\} \subset \{1, \dots, n\}$ . Suppose that  $K_i$  is diffeomorphic to  $K'_i$ . If  $\text{alk}(K_i, K_j) = \text{alk}(K'_i, K'_j)$  for all distinct  $i, j$ , we say that  $\mathcal{K}$  and  $\mathcal{K}'$  are *alinking-equivalent*. If there is a surface-link  $\mathcal{K}'' = (K''_1, \dots, K''_n)$  obtained from  $\mathcal{K}' = (K'_1, \dots, K'_n)$  by changing orders of components such that  $\mathcal{K}$  and  $\mathcal{K}''$  are alinking-equivalent and such that for each  $i \in \{1, \dots, n\}$ ,  $K_i$  is diffeomorphic to  $K''_i$ , then we say that  $\mathcal{K}$  and  $\mathcal{K}'$  are *weakly alinking-equivalent*.

We have the following.

**Theorem 7.2.** (See [3, Lemma 4.2] and [5, the first several lines of page 523].)

- (1) Let  $J$  be a spherical 2-knot  $\subset S^4$ . Let  $p$  be a prime power. Take the  $p$ -fold branched cyclic covering space of  $S^4$  along  $J$ , and call it  $S'$ . Then  $S'$  is a  $\mathbb{Z}_p$ -homology sphere.
- (2) Take  $S^4 \times [0, 1]$ . Let  $i \in \{0, 1\}$ . Let  $J_i$  be a spherical 2-knot contained in  $S^4 \times \{i\}$ . Let  $p$  be a prime power. Take a submanifold  $X \subset S^4 \times [0, 1]$  which gives cobordism between  $J_0$  and  $J_1$ . (Note: [7] ensures that the existence of  $X$ ). Take the  $p$ -fold branched cyclic covering space of  $S^4 \times [0, 1]$  along  $X$ , and call it  $M$ . Let  $*$   $\in \mathbb{Z}$ . Then

$$H_*(M; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } * = 0, 4 \\ 0 & \text{else} \end{cases}.$$

**Definition 7.3.** Let  $E = (J, K_1, \dots, K_m)$  be a surface-link of the standard 4-sphere  $S^4$ . Let  $n \in \mathbb{N}$ . Let  $M$  be the  $n$ -fold branched cyclic covering space of  $S^4$  along  $J$ . Take the lift  $\tilde{K}$  of the sublink  $K = (K_1, \dots, K_m)$  associated with this branched cyclic covering. This submanifold  $\tilde{K} \subset M$  is called the  *$n$ -covering-link of  $E$  along  $J$* .

**Proposition 7.4.** Take  $J, E, \tilde{K}$  in Definition 7.3. Suppose that  $J$  is the trivial spherical 2-knot and that  $E$  is semi-boundary. Then  $\tilde{K}$  is contained in the standard 4-sphere, and is an  $m \cdot n$ -component surface-link.

**Proof of Proposition 7.4.** Since  $J$  is the trivial spherical 2-knot, the  $n$ -fold branched cyclic covering space of  $S^4$  along  $J$  is the standard 4-sphere. Let  $N(J)$  be the tubular neighborhood of  $J$  in  $S^4$ . Take a  $\mathbb{Z}_n$ -covering space of  $S^4 - \text{Int}N(J)$  associated with this branched cyclic covering. This  $\mathbb{Z}_n$ -covering space of  $S^4 - \text{Int}N(J)$  is regarded as the total space of a  $\mathbb{Z}_n$ -fiber bundle over  $S^4 - \text{Int}N(J)$ . Since any 1-cycle in  $K$  is null-homologous in  $S^4 - \text{Int}N(J)$ , the restriction of this  $\mathbb{Z}_n$ -fiber space to  $K$  is the trivial bundle. Hence Proposition 7.4 holds.  $\square$

It is convenient to define the following terminologies.

**Definition 7.5.** Let  $*$   $\in \{0, 1\}$ . Let  $L_*$  be a surface-link contained in the standard 4-sphere  $S^4 \times \{*\}$ . Let  $\mathcal{M}$  be a connected compact oriented 5-manifold which is not necessarily diffeomorphic to  $S^4 \times [0, 1]$ . Suppose that  $\partial\mathcal{M}$  is the disjoint union of two

copies of the standard 4-sphere. Call one  $S^4 \times \{0\}$ , and the other  $S^4 \times \{1\}$ . In Definition 1.1, replace  $S^4 \times [0, 1]$  with  $\mathcal{M}$ . We say that  $L_0$  and  $L_1$  are *cobordant in  $\mathcal{M}$* . We say that an embedding map  $f$  (resp. a submanifold  $f((F_1 \times [0, 1]) \amalg \dots \amalg (F_\mu \times [0, 1])) \subset \mathcal{M}$ ) gives *cobordism between  $L_0$  and  $L_1$  in  $\mathcal{M}$* .

**Theorem 7.6.** *Let  $\iota \in \{0, 1\}$ . Let  $E_\iota = (K_{\iota 0}, K_{\iota 1}, \dots, K_{\iota, m})$  be a semi-boundary surface-link such that  $K_{\iota 0}$  is the spherical trivial 2-knot. Let  $\ast \in \{0, 1, \dots, m\}$ . Suppose that  $E_1$  is cobordant to  $E_2$ , and that  $C = C_0 \amalg C_1 \amalg \dots \amalg C_m \subset S^4 \times [0, 1]$  (resp.  $C_\ast$ ) gives cobordism between  $E_0$  and  $E_1$  (resp.  $K_{0\ast}$  and  $K_{1\ast}$ ). Let  $p$  be a prime power. Take the  $p$ -fold branched cyclic covering space,  $\mathcal{M}$ , of  $S^4 \times [0, 1]$  along  $C$ . Thus we obtain the  $p$ -covering-link  $\mathcal{E}_\iota$  of  $E_\iota$  along  $K_{\iota 0}$ . Then  $\mathcal{E}_\iota$  is contained in the standard 4-sphere and the alinking number associated with the components of  $\mathcal{E}_\iota$  makes sense. Then  $\mathcal{E}_0$  is weakly alinking-equivalent to  $\mathcal{E}_1$ .*

**Proof of Theorem 7.6.** Let  $\iota \in \{0, 1\}$ . Since  $E_\iota$  is a semi-boundary surface-link and  $K_{\iota 0}$  is the trivial spherical knot,  $\mathcal{E}_\iota$  is a  $p \cdot m$ -component surface-link by Proposition 7.4, and let  $\mathcal{E}_\iota = (\mathcal{E}_{\iota 1}, \dots, \mathcal{E}_{\iota, p \cdot m})$ . Let  $\xi \in \{1, \dots, m\}$ . Furthermore the lift of  $C_\xi$  is a disjoint union of compact connected components  $C_{\xi 1} \amalg, \dots, \amalg C_{\xi, p}$ . We can give an order to all elements of  $\{C_{\xi, \#} \mid \xi = 1, \dots, m, \# = 1, \dots, p\}$ , and call them  $A_1, \dots, A_{p \cdot m}$ . Let  $\natural \in \{1, \dots, p \cdot m\}$ . Let  $\partial A_\natural = \mathcal{E}_{0\natural} \amalg \mathcal{E}_{1\natural}$ . Note that  $\mathcal{E}_{0\natural}$  is diffeomorphic to  $\mathcal{E}_{1\natural}$  and that  $A_\natural$  is diffeomorphic to  $\mathcal{E}_{0\natural} \times [0, 1]$ . Thus  $A_1 \amalg \dots \amalg A_{p \cdot m}$  (resp  $A_\natural$ ) gives cobordism between  $\mathcal{E}_0$  and  $\mathcal{E}_1$  (resp.  $\mathcal{E}_{0\natural}$  and  $\mathcal{E}_{1\natural}$ ) in  $\mathcal{M}$ . Let  $\eta \in \mathbb{Z}$ . By Theorem 7.4,  $\mathcal{M}$  satisfies the condition  $H_\eta(\mathcal{M}; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } \eta = 0, 4 \\ 0 & \text{else} \end{cases}$ . Since  $K_{01}$  and  $K_{11}$  are the trivial spherical 2-knot,  $\partial \mathcal{M}$  is the disjoint union of two copies of the standard 4-sphere. Call one which include  $\mathcal{E}_0$ ,  $S_0$ , and the other which include  $\mathcal{E}_1$ ,  $S_1$ .

Let  $\{i, j\} \subset \{1, \dots, p \cdot m\}$ . It suffices to prove that for any distinct  $i, j$ , the alinking number  $\text{alk}(\mathcal{E}_{0j} \subset \mathcal{E}_0, \mathcal{E}_{0i} \subset \mathcal{E}_0)$  of  $\mathcal{E}_{0j}$  in  $\mathcal{E}_0$  around  $\mathcal{E}_{0i}$  in  $\mathcal{E}_0$  is equivalent to the alinking number  $\text{alk}(\mathcal{E}_{1j} \subset \mathcal{E}_1, \mathcal{E}_{1i} \subset \mathcal{E}_1)$  of  $\mathcal{E}_{1j}$  in  $\mathcal{E}_1$  around  $\mathcal{E}_{1i}$  in  $\mathcal{E}_1$ . Let  $\iota \in \{0, 1\}$ . Take a Seifert hypersurface  $V_{\iota, i}$  for  $\mathcal{E}_{\iota, i}$ . Of course  $V_{\iota, i} \cap \mathcal{E}_{\iota, j} \neq \emptyset$  may hold. Note that  $V_{0i} \cup A_i \cup V_{1i}$  is a closed oriented 3-manifold and a  $\mathbb{Z}$ -3-cycle in  $\mathcal{M}$ . Let  $\tau \in \mathbb{Z}$ . Since  $H_\tau(\mathcal{M}; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } \tau = 0, 4 \\ 0 & \text{else} \end{cases}$ , there is a natural number  $y_i$  and a  $\mathbb{Z}$ -4-chain  $\Gamma$  such that

$\partial \Gamma = y_i \cdot (V_{0i} \cup A_i \cup V_{1i})$ , where  $\partial$  denotes the boundary of a chain (not a manifold). Furthermore we can take an immersion map  $g_i$  from a compact oriented 4-manifold  $X_i$  to  $\mathcal{M}$  with the following properties:  $g_i|_{\text{Int} X_i}$  is an embedding map.  $g_i(\partial X_i) = V_{0i} \cup A_i \cup V_{1i}$ .  $g_i|_{\partial X_i} : \partial X_i \rightarrow V_{0i} \cup A_i \cup V_{1i}$  defines a  $\mathbb{Z}_{y_i}$ -fiber bundle. (*Reason.* Use Thom-Pontrjagin construction.)



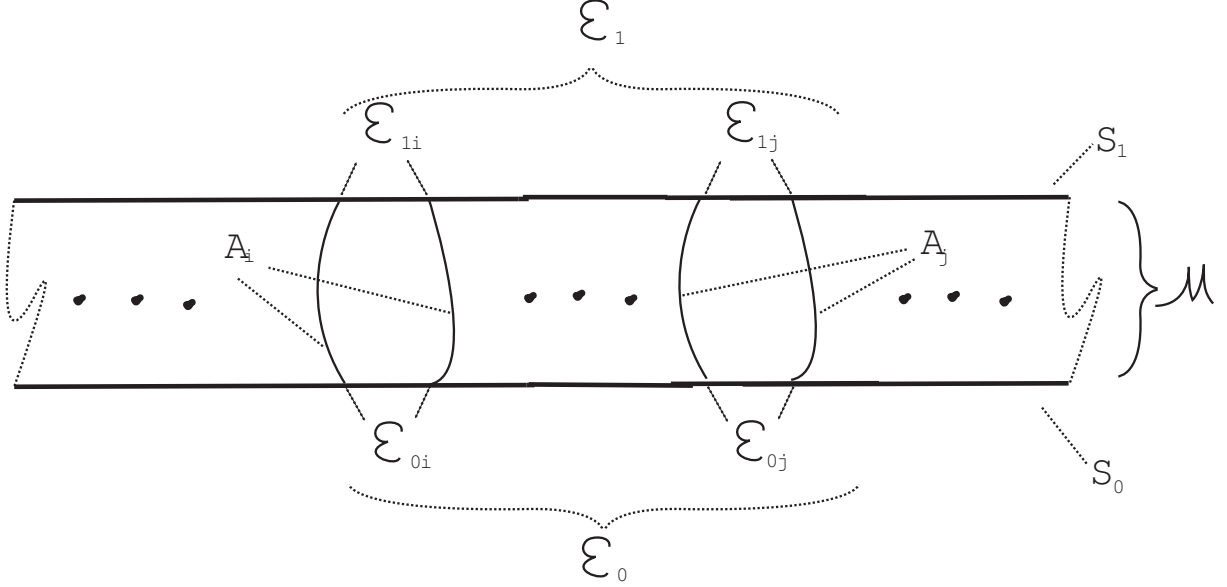


FIGURE 7.1.  $\mathcal{M}$ .

Take any embedded circle  $C$  in  $\mathcal{E}_{0j}$ . Take an embedded circle  $C'$  in  $\mathcal{E}_{1j}$  with the following properties: There is  $Q$  embedded in  $A_j$ .  $Q$  is diffeomorphic to  $S^1 \times [0, 1]$ .  $\partial Q = C \amalg C'$ , where  $\partial$  denotes the boundary of a manifold.

Consider  $Q \cap X_i$ . Then  $Q \cap X_i$  is a 1-chain. The boundary of the 1-chain is algebraically zero points. Since  $A_i \cap A_j = \emptyset$ , the points, which make the boundary of the 1-chain, are included in  $V_{0i} \amalg V_{1i}$ . Hence  $V_{0i} \cap \mathcal{E}_{0j}$  and  $V_{1i} \cap \mathcal{E}_{1j}$  are algebraically the same number of oriented 0-cells. Note that the absolute value of the algebraic number  $V_{0i} \cap \mathcal{E}_{0j}$  (resp.  $V_{1i} \cap \mathcal{E}_{1j}$ ) is  $y_i \cdot \text{alk}(\mathcal{E}_{0j} \subset \mathcal{E}_0, \mathcal{E}_{0i} \subset \mathcal{E}_0)$  (resp.  $y_i \cdot \text{alk}(\mathcal{E}_{1j} \subset \mathcal{E}_1, \mathcal{E}_{1i} \subset \mathcal{E}_1)$ ). Therefore  $\text{alk}(\mathcal{E}_{0j} \subset \mathcal{E}_0, \mathcal{E}_{0i} \subset \mathcal{E}_0) = \text{alk}(\mathcal{E}_{1j} \subset \mathcal{E}_1, \mathcal{E}_{1i} \subset \mathcal{E}_1)$ .  $\square$

**Note.** In Theorem 7.6, if we check the covering-links more elaborately, we can prove a stronger condition on which pair of components of  $\mathcal{E}_0$  and which pair of components of  $\mathcal{E}_1$  are equivalent. (The way is written implicitly in this section). However we do not need it in order to prove our results in this paper. So we do not discuss it further here.

## 8. PROOF OF MAIN THEOREM 1.4

Make a spun surface-link from  $A \amalg B \amalg C$  in Figure 8.1. In Figure 8.1 we draw the  $xy$ -plane and omit the  $x$ -,  $y$ -,  $z$ -axes. From now we do like this when we draw this kind of figures. Let  $\mathcal{L} = (P, Q, R)$  be the resultant surface-link such that  $P$  (resp.  $Q, R$ ) is

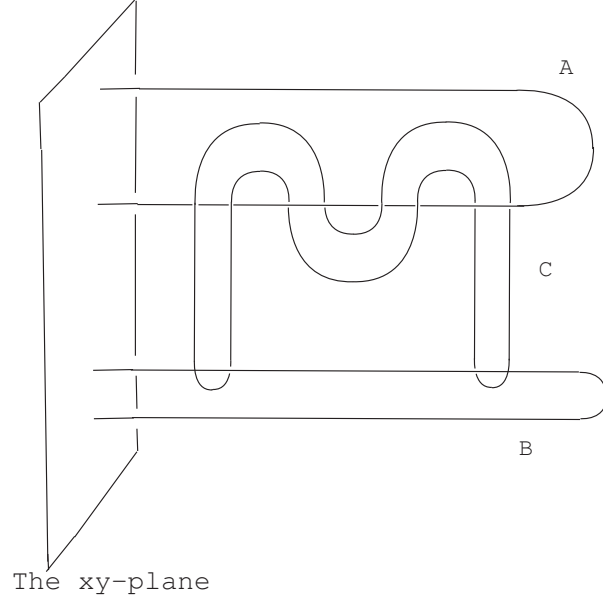


FIGURE 8.1. **Make a Spun-surface-link from this.**

made from  $A$  (resp.  $B$ ,  $C$ ). Let  $\mathcal{L}' = (P', Q', R')$  be the resultant surface-link such that  $P'$  (resp.  $Q'$ ,  $R'$ ) is made from  $B$  (resp.  $A$ ,  $C$ ). Thus we have the following.

**Fact 8.1.** *There is an orientation-preserving diffeomorphism map  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  (or  $S^4 \rightarrow S^4$ ) such that  $f|_{\mathcal{L}}$  is an order-nonpreserving orientation-preserving diffeomorphism map  $\mathcal{L} \rightarrow \mathcal{L}'$ .*

By the construction, we have the following.

**Fact 8.2.** *All sublinks of  $\mathcal{L}$  (resp.  $\mathcal{L}'$ ) are the standard links. (Recall that we defined that  $\mathcal{L}$  (resp.  $\mathcal{L}'$ ) itself and the empty set are not sublinks.)*

Fact 8.2 implies Main Theorem 1.4.(1), (2)(ii), (3), (4).(ii), and (5).(ii).

Fact 8.1 implies Main Theorem 1.4.(2)(i), and (4)(i), and (5).(i).

Fact 8.1 and Theorem 6.5 imply Main Theorem 1.4.(5).(ii).

We prove Main Theorem 1.4.(6). Let  $p$  be a sufficiently large prime power. Take the  $p$ -covering-link  $\mathcal{Z}$  (resp.  $\mathcal{Z}'$ ) of  $\mathcal{L}$  (resp.  $\mathcal{L}'$ ) along  $P$  (resp.  $P'$ ).

By the definition of  $\mathcal{L}$ , we have the following: Let  $\mathcal{Z}_i$  and  $\mathcal{Z}_j$  be different components of  $\mathcal{Z}$ . The alinking number of  $\mathcal{Z}_i$  around  $\mathcal{Z}_j$  is one or zero.

By the definition of  $\mathcal{L}'$ , we have the following: There are two different components  $\mathcal{Z}'_k$  and  $\mathcal{Z}'_l$  of  $\mathcal{Z}'$  such that the alinking number of  $\mathcal{Z}'_k$  around  $\mathcal{Z}'_l$  is two.

Therefore  $\mathcal{Z}$  and  $\mathcal{Z}'$  are not weakly alinking-equivalent. By Theorem 7.6,  $\mathcal{L}$  and  $\mathcal{L}'$  are not cobordant.

This completes the proof of Main Theorem 1.4.  $\square$

## 9. A THEOREM

We give another partial solution to Problem 1.3. Theorem 6.2 lets us naturally formulate the following problem, which is included in Problem 1.3.

**Problem 9.1.** Find a new  $(S^2, T^2)$ -link which is not cobordant to the standard link.

Theorem 9.2 is a partial solution to this problem since  $\mathcal{L}^{(0)}$  in Proof of Theorem 9.2 is the standard link.

By Theorem 6.2 we have the following: if we replace ‘the standard link’ with ‘a boundary link’ in Problem 9.1, the new one is the same as the old one. By Corollary 2.2 we have the following: if we replace ‘ $(S^2, T^2)$ -link’ with ‘semi-boundary  $(S^2, T^2)$ -link’ in Problem 9.1, the new one is the same as the old one. If we replace  $(S^2, T^2)$  with  $(S^2, S^2)$  in Problem 9.1, it is Problem 1.2<sup>1</sup>.

**Theorem 9.2.** *There is a set  $\{(S^2, T^2)\text{-links } \mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)}) | i, j \in \mathbb{N} \cup \{0\}, \mathcal{L}^{(i)} \text{ is non-cobordant to } \mathcal{L}^{(j)} \text{ for any distinct } i, j\}$  with the following properties:*

- (1) *The alinking number of  $\mathcal{L}^{(i)}$  is the same as that of  $\mathcal{L}^{(j)}$  for any distinct  $i, j$ .*
- (2) *The Sato-Levine invariant of  $\mathcal{L}^{(i)}$  is the same as that of  $\mathcal{L}^{(j)}$  for any distinct  $i, j$ .*
- (3) *The Cochran sequence of  $\mathcal{L}^{(i)}$  is the same as that of  $\mathcal{L}^{(j)}$  for any distinct  $i, j$ .*
- (4) *If there is an element  $* \in \mathbb{N}_{\geq 2} \cup \{\omega\}$  such that for any distinct  $i, j$ , we can define the Orr invariant  $\theta_*(\mathcal{L}^{(i)}, \tau^{(i)})$  for a meridian  $\tau^{(i)}$  and  $\theta_*(\mathcal{L}^{(j)}, \tau^{(j)})$  for a meridian  $\tau^{(j)}$ , then  $\theta_*(\mathcal{L}^{(i)}) = \theta_*(\mathcal{L}^{(j)}) = 0$ .*
- (5) *Our new way in §7 implies that  $\mathcal{L}^{(i)}$  is non-cobordant to  $\mathcal{L}^{(j)}$  for any distinct  $i, j$ .*

**Proof of Theorem 9.2.** Make a spun  $(S^2, T^2)$ -link  $\mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)})$  (resp.  $\mathcal{J}^{(i)}, \mathcal{K}^{(i)}$ ) from  $P \amalg Q$  (resp.  $P, Q$ ) in Figure 9.1 by the rotation around the  $xy$ -plane as the axis. Figure 9.2 is the  $i = 1$  case of Figure 9.1. We prove that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies the condition (1)-(5) of Theorem 9.2. By the construction  $\mathcal{J}^{(i)}$  is the trivial spherical 2-knot for each  $i$ .

We prove that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(1). Take a 1-link  $(P \cup I, Q)$  in  $\{(x, y, z) | z \geq 0\}$  as in Figure 9.3. Note that there is a Seifert surface  $V$  (resp.  $W$ ) for

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<sup>1</sup> Note the following facts written in [5]: A natural ‘even dimensional spherical link’ version of Problem 1.2 is ‘Are all even-dimensional spherical links slice?’ If  $n \geq 3$ , a natural ‘spherical  $n$ -link’ version of Problem 1.2 is ‘Are all spherical  $n$ -links cobordant to boundary links?’ Note that the latter problem includes the former one. The latter one and Theorem 6.2 let us naturally regard Problem 9.1 as a surface-link version of Problem 1.2.

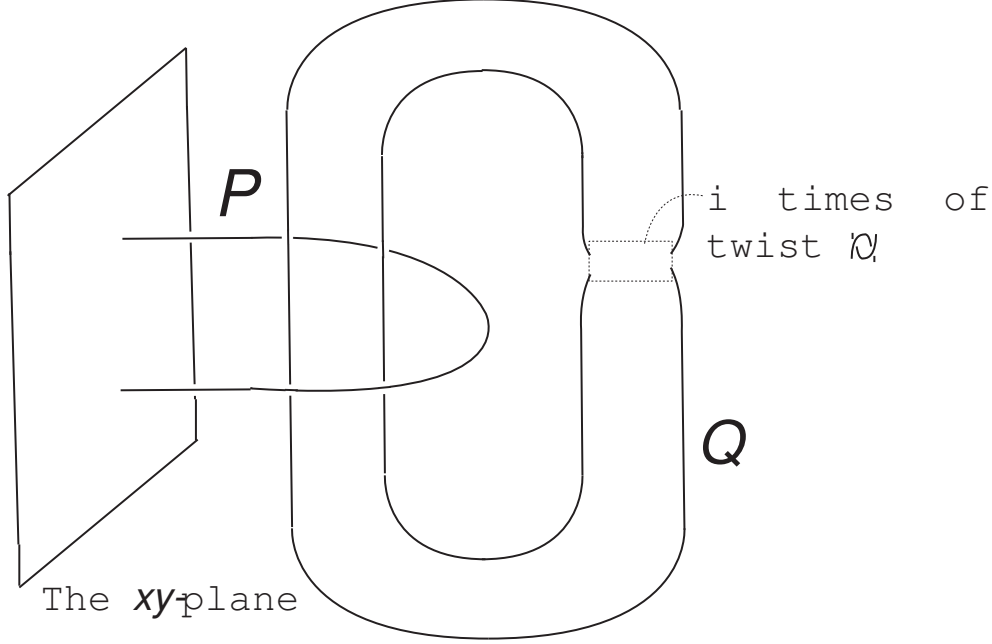


FIGURE 9.1.  $P \amalg Q$  (resp.  $P, Q$ ) is made into the spun  $(S^2, T^2)$ -link  $\mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)})$  (resp.  $\mathcal{J}^{(i)}, \mathcal{K}^{(i)}$ ) by the rotation.

$P \cup I$  (resp.  $Q$ ) such that  $V \cap Q = W \cap (P \cup I) = \phi$ . Note that  $V \cup W \subset \{(x, y, z) | z \geq 0\}$ . We can suppose that  $V \cap \{(x, y, z) | z = 0\} = I$  and  $W \cap \{(x, y, z) | z = 0\} = \phi$ . When we rotate  $P \amalg Q$  and make  $\mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)})$ , rotate  $V$  (resp.  $W$ ) together. Then the result  $\mathcal{V}^{(i)}$  (resp.  $\mathcal{W}^{(i)}$ ) is a Seifert hypersurface for  $\mathcal{J}^{(i)}$  (resp.  $\mathcal{K}^{(i)}$ ) such that  $\mathcal{V}^{(i)} \cap \mathcal{K}^{(i)} = \mathcal{W}^{(i)} \cap \mathcal{J}^{(i)} = \phi$ . The existence of  $\mathcal{V}^{(i)}$  implies that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(1). We do not use  $\mathcal{W}^{(i)}$  here but we will use it from now.

We prove that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(2). By Theorem 9.2.(1), we can define the Sato-Levine invariant for  $\mathcal{L}^{(i)}$ . We can suppose that  $V$  intersects  $W$  transversely and that  $V \cap W$  is  $R$  in Figure 9.4. We can suppose that  $\mathcal{V}^{(i)}$  intersect  $\mathcal{W}^{(i)}$  transversely and that  $\mathcal{V}^{(i)} \cap \mathcal{W}^{(i)}$  is the result  $\mathcal{T}^{(i)}$  of rotating  $R$  in Figure 9.4 around the  $xy$  plane when we rotate  $P \amalg Q$  and make  $\mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)})$ . Note that  $\mathcal{T}^{(i)}$  is diffeomorphic to the torus. Take a point in  $R$ . The point becomes a circle when we make  $R$  into  $\mathcal{T}^{(i)}$  by the rotation. Let  $\sigma$  be the spin structure on the circle which is defined by  $\mathcal{V}^{(i)}$  and  $\mathcal{W}^{(i)}$ . Then  $[S^1, \sigma] = 0 \in \Omega_1^{\text{Spin}}$ . Hence we have the following: Let  $\tau$  be the spin structure on  $\mathcal{T}^{(i)}$  which is defined by  $\mathcal{V}^{(i)}$  and  $\mathcal{W}^{(i)}$ . Then  $[T^2, \tau] = 0 \in \Omega_2^{\text{Spin}}$ . See [6, §5.6] and [8, §IV] for the spin structure. Since the Sato-Levine invariant of  $\mathcal{L}^{(i)}$  is  $[T^2, \tau] \in \Omega_2^{\text{Spin}} \cong \mathbb{Z}_2$

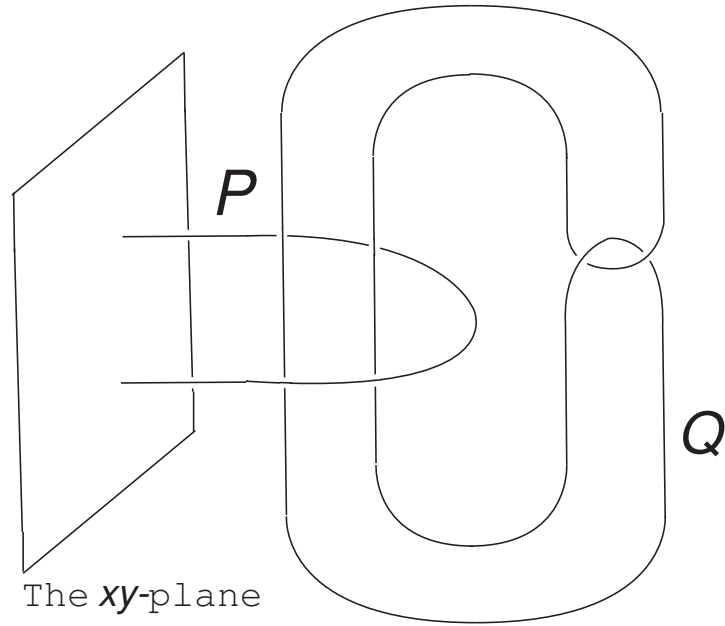


FIGURE 9.2. The  $i = 1$  case of Figure 9.1, that is,  $\mathcal{L}^{(1)}$

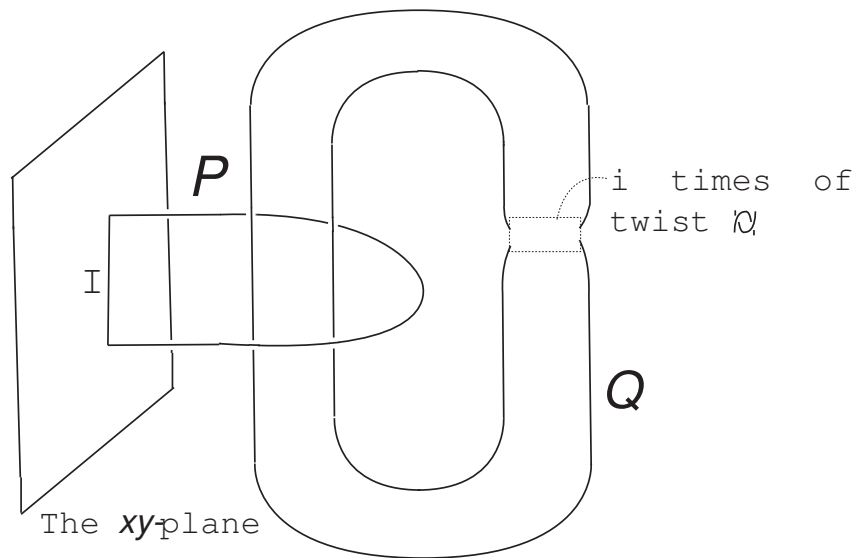


FIGURE 9.3. We add  $I$  to Figure 9.1

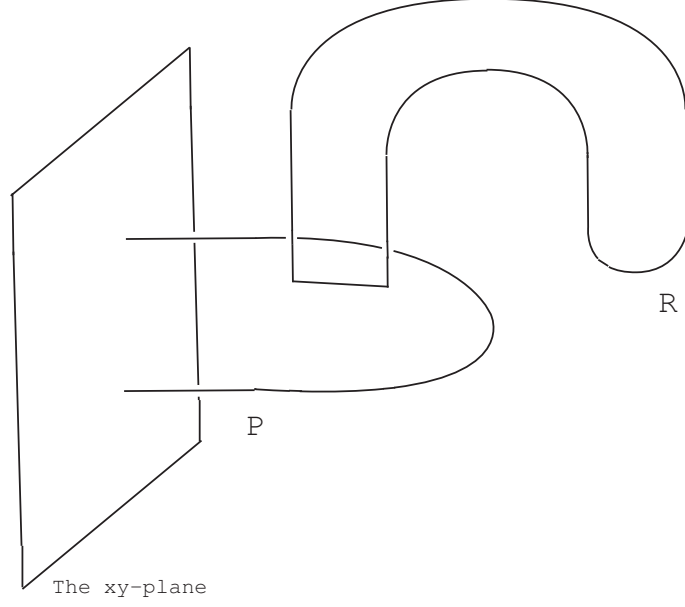


FIGURE 9.4.  $P \amalg R$  is a part of a 1-link whose spun link is the derivative of  $\mathcal{L}^{(i)}$ .

(see [10]), the Sato-Levine invariant of  $\mathcal{L}^{(i)}$  is zero. Hence  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(2).

We prove that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(3). By Theorem 9.2.(1), we can define the Cochran sequence for  $\mathcal{L}^{(i)}$ . The derivative  $D(\mathcal{L}^{(i)})$  of  $\mathcal{L}^{(i)}$  is the spun  $(S^2, T^2)$ -link of Figure 9.4. It is the standard link. Hence  $\beta^j(\mathcal{L}^{(i)}) = 0$  if  $j \geq 2$ . By this fact and Theorem 9.2.(2), we have  $\{\beta^j(\mathcal{L}^{(i)})\}_{j \in \mathbb{N}}$  is trivial. Hence  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(3).

By Theorem 6.5,  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(4).

We prove that  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(5). Let  $p$  be a prime power. By Proposition 7.4, the  $p$ -covering-link  $\mathcal{L}^{(i,p)}$  of  $\mathcal{L}^{(p)}$  is a  $(\underbrace{T^2, \dots, T^2}_p)$ -link  $(\mathcal{K}_1^{(i,p)}, \dots, \mathcal{K}_p^{(i,p)})$ .

Figure 9.5 is the case  $i = 1$  and  $p = 3$ . Let  $\nu \in \{1, 2, 3\}$ . Note that  $K_\nu$  in Figure 9.5 is made into  $\mathcal{K}_\nu^{(i,p)}$ .

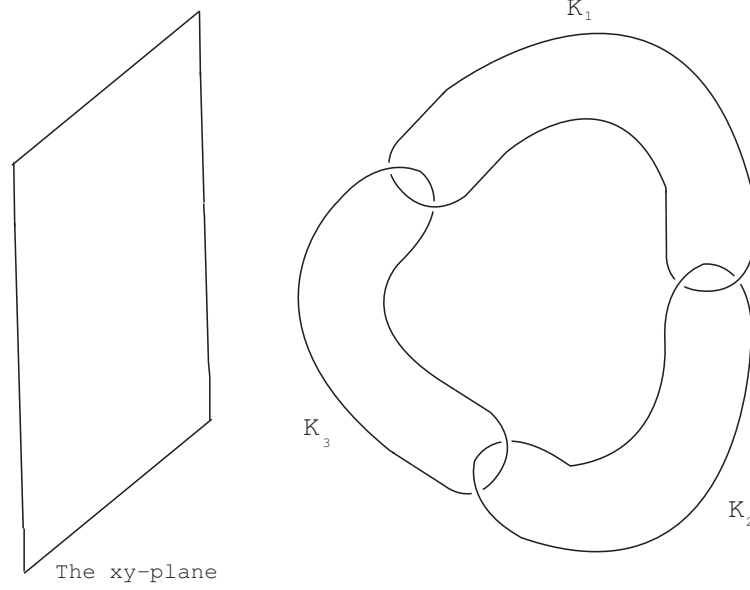


FIGURE 9.5. A 1-link  $(K_1, K_2, K_3)$  whose spun  $(S^2, T^2)$ -link is the 3-covering-link  $\mathcal{L}^{(1,3)} = (\mathcal{K}_1^{(1,3)}, \mathcal{K}_2^{(1,3)}, \mathcal{K}_3^{(1,3)})$  of  $\mathcal{L}^{(1)}$ .  $K_i$  is made into  $\mathcal{K}_i^{(1,3)}$  by the rotation.

By the construction of  $\mathcal{L}^{(i,p)}$ , we have the following: If  $p$  is sufficient large, then for any distinct  $\alpha, \beta \in \{1, \dots, p\}$ , the alinking number of  $\mathcal{K}_\alpha^{(i,p)}$  around  $\mathcal{K}_\beta^{(i,p)}$  is  $i$  or  $0$ . By Theorem 7.6,  $\mathcal{L}^{(i)}$  is not cobordant to  $\mathcal{L}^{(j)}$  for any distinct  $i, j \in \mathbb{N} \cup \{0\}$ . Hence  $\{\mathcal{L}^{(i)} | i \in \mathbb{N} \cup \{0\}\}$  satisfies Theorem 9.2.(5).

This completes the proof of Theorem 9.2.  $\square$

**Note.** (1) By using Theorem 6.5, the author could make infinitely many examples like the set  $\{\mathcal{L}^i\}$  in Theorem 9.2.

(2) The author could prove that we can define the Cochran sequence for the spun  $(S^2, T^2)$ -link of any semi-boundary 2-component 1-link, where

(any component of the 1-link)  $\cap$  (the axis  $\mathbb{R}^3$ ) is  $\phi$  or the interval, and that it is trivial. In order to prove this fact, we use the  $T_{bd}^2$  spin structure (see [8]) as in the proof of Theorem 9.2.

## 10. DISCUSSION

Let  $i \in \mathbb{N} \cup \{0\}$ . Take the  $(S^2, T^2)$ -link  $\mathcal{L}^{(i)} = (\mathcal{J}^{(i)}, \mathcal{K}^{(i)}) \subset S^4$  in Proof of Theorem 9.2. Suppose that  $\mathcal{L}^{(i)} \subset B^4 \subset S^4$ . Take the standard  $T^2$ -knot  $G \subset S^4 - B^4$ . By using an

embedded 3-dimensional 1-handle  $\subset S^4$ , connect  $\mathcal{J}^{(i)}$  and  $G$ . Thus we obtain a  $(T^2, T^2)$ -link  $\mathcal{M}^{(i)} = (\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$  by this connected-sum. Let  $\mathcal{M}'^{(i)} = (\mathcal{E}'^{(i)}, \mathcal{F}'^{(i)})$  be a  $(T^2, T^2)$ -link which is made from  $\mathcal{M}^{(i)}$  by changing the order. We ask a question.

**Question 10.1.** Let  $i \in \mathbb{N}$ . Are the above  $\mathcal{M}^{(i)}$  and  $\mathcal{M}'^{(i)}$  cobordant?

Let  $i \in \mathbb{N}$ . By Theorem 9.2 and the construction of  $\mathcal{M}^{(i)}$  and  $\mathcal{M}'^{(i)}$ , we have the following:

- (1)  $\text{alk}(\mathcal{E}^{(i)} \subset \mathcal{M}^{(i)}, \mathcal{F}^{(i)} \subset \mathcal{M}^{(i)}) = \text{alk}(\mathcal{E}'^{(i)} \subset \mathcal{M}'^{(i)}, \mathcal{F}'^{(i)} \subset \mathcal{M}'^{(i)})$   
 $\text{alk}(\mathcal{F}^{(i)} \subset \mathcal{M}^{(i)}, \mathcal{E}^{(i)} \subset \mathcal{M}^{(i)}) = \text{alk}(\mathcal{F}'^{(i)} \subset \mathcal{M}'^{(i)}, \mathcal{E}'^{(i)} \subset \mathcal{M}'^{(i)})$
  - (2) The 2-component-Sato-Levine invariant of  $\mathcal{M}^{(i)}$  is the same as that of  $\mathcal{M}'^{(i)}$ .
  - (3) The Cochran sequence of  $\mathcal{M}^{(i)}$  is the same as that of  $\mathcal{M}'^{(i)}$ .
  - (4)  $\pi_{\mathcal{M}^{(i)}}$  and  $\pi_{\mathcal{M}'^{(i)}}$  are Stallings-equivalent.
  - (5) Let  $\# \in \mathbb{N}_{\geq 2} \cup \{\omega\}$ . The following two are equivalent.
    - (I) We can define the Orr invariant  $\theta_{\#}(\mathcal{M}^{(i)}, \tau)$  for a meridian  $\tau$
    - (II) We can define  $\theta_{\#}(\mathcal{M}'^{(i)}, \tau')$  for a meridian  $\tau'$ .
- Suppose that (I) (resp. (II)) holds. Then  $\theta_*(\mathcal{M}^{(i)}) = \theta_*(\mathcal{M}'^{(i)}) = 0$ .

However we cannot use the alinking number associated with covering-links in general as we can in Theorem 7.6. *Reason.* The covering space of  $S^4$  along  $\mathcal{E}'^{(i)}$  (resp.  $\mathcal{F}'^{(i)}, \mathcal{E}^{(i)}, \mathcal{F}^{(i)}$ ) is not an integral homology sphere.

Even if we can define the alinking number associated with covering-links in some special cases, it may not be a cobordism invariant. *Reason:* We omit  $(i)$  in  $\mathcal{M}^{(i)} = (\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$  and  $\mathcal{M}'^{(i)} = (\mathcal{E}'^{(i)}, \mathcal{F}'^{(i)})$  from now for the convenience. Suppose that there is a submanifold  $E \amalg F \subset S^4$  which gives cobordism between  $\mathcal{M}$  and  $\mathcal{M}'$ . Note that  $E$  (resp.  $F$ ) is diffeomorphic to  $T^2 \times [0, 1]$ . Suppose that  $E$  (resp.  $F$ ) gives cobordism between  $\mathcal{E}$  and  $\mathcal{E}'$  (resp.  $\mathcal{F}$  and  $\mathcal{F}'$ ). The covering space  $X$  of  $S^4 \times [0, 1]$  along  $E$  (resp.  $F$ ) does not satisfy the condition  $H_i(X, \mathbb{Z}) \cong H_i(S^4, \mathbb{Z})$  for all  $i$ . The way in Proof of Theorem 7.6 cannot be used.

Recall that the above (4) follows from [17, 5.2 Theorem]. By the way, [17, 5.2 Theorem] follows from [17, 5.1 Theorem]. We may be able to prove that  $\mathcal{M}$  and  $\mathcal{M}'$  are non-cobordant if we use [17, 5.1 Theorem] as follows. Note that the tubular neighborhood of  $\mathcal{M}$  in  $S^4$  is diffeomorphic to  $\mathcal{M} \times D^2$ , and call it  $\mathcal{M} \times D^2$ . By [17, 5.1 Theorem], for each  $k \in \mathbb{N}$ , natural inclusion maps induce the following isomorphisms.

$$\begin{aligned} & \pi_1(S^4 - \text{Int}\mathcal{M} \times D^2) / (\pi_1(S^4 - \text{Int}\mathcal{M} \times D^2))_k \\ & \quad \downarrow \alpha, \cong \\ & \pi_1(S^4 \times [0, 1] - \text{Int}(E \amalg F) \times D^2) / (\pi_1(S^4 \times [0, 1] - \text{Int}(E \amalg F) \times D^2))_k \end{aligned}$$



$$\begin{aligned} & \pi_1(S^4 - \text{Int}\mathcal{M}' \times D^2) / (\pi_1(S^4 - \text{Int}\mathcal{M}' \times D^2))_k \\ & \quad \downarrow \beta, \cong \\ & \pi_1(S^4 \times [0, 1] - \text{Int}(E \amalg F) \times D^2) / (\pi_1(S^4 \times [0, 1] - \text{Int}(E \amalg F) \times D^2))_k \end{aligned}$$

These give an isomorphism.

$$\begin{aligned} & \pi_1(S^4 - \text{Int}\mathcal{M} \times D^2) / (\pi_1(S^4 - \text{Int}\mathcal{M} \times D^2))_k \\ & \quad \downarrow \gamma, \cong \\ & \pi_1(S^4 - \text{Int}\mathcal{M}' \times D^2) / (\pi_1(S^4 - \text{Int}\mathcal{M}' \times D^2))_k \end{aligned}$$

By the construction of  $\mathcal{M}$  and  $\mathcal{M}'$ , there is an orientation preserving diffeomorphism  $f : S^4 \rightarrow S^4$  such that  $f_{\mathcal{M}}$  is an orientation-preserving, order-nonpreserving diffeomorphism  $\mathcal{M} \rightarrow \mathcal{M}'$ . This gives another isomorphism.

$$\pi_1(S^4 - \text{Int}\mathcal{M} \times D^2) \xrightarrow{\delta, \cong} \pi_1(S^4 - \text{Int}\mathcal{M}' \times D^2)$$

The combination of these isomorphisms,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  may give a restriction to a link-type of  $\mathcal{M}$ .

By using the above way, we may prove that  $\mathcal{L}$  and  $\mathcal{L}'$  in Main theorem 1.4 are non-cobordant without using Main theorem 1.4(6).

By using [17, 5.2 Theorem], for a pair  $i, j$ , we may prove that  $\mathcal{L}^{(i)}$  and  $\mathcal{L}'^{(i)}$  in Theorem 9.2 are noncobordant without using Theorem 9.2 (5). (Note. It may be proved as follows.  $\mathcal{L}^{(i)}$  and  $\mathcal{L}^{(j)}$  may not be Stallings-equivalent. Use this fact.)

However, if we can do these, they must be more complicated than the way in this paper.

We give another cobordism invariant. We use the notations in Theorem 7.6 and Proof of Theorem 7.6. Take the cobordant surface-links  $E_1$  and  $E_2$ . Let  $r$  be a sufficiently large natural number. Let  $p$  be  $2^r$ . For any distinct  $i, j$ , take  $\mathcal{E}_{0i}$  and  $\mathcal{E}_{0j}$ . By Theorem 2.3, the following two are equivalent.

(I) The alinking number  $\mathcal{E}_{0i}$  (resp.  $\mathcal{E}_{0j}$ ) around  $\mathcal{E}_{0j}$  (resp.  $\mathcal{E}_{0i}$ ) is zero.

(II) The alinking number  $\mathcal{E}_{1i}$  (resp.  $\mathcal{E}_{1j}$ ) around  $\mathcal{E}_{1j}$  (resp.  $\mathcal{E}_{1i}$ ) is zero.

We suppose (I) (resp. (II)). Let  $*$   $\in$   $\{0, 1\}$ . Hence we can suppose that  $V_{*i}$  and  $V_{*j}$  are special Seifert hypersurface. Let  $F_*$  be a closed oriented 2-dimensional submanifold  $V_{*i} \cap V_{*j}$ . We induce the spin structure  $\phi_*$  on  $F_*$  by using  $V_{*i}$ , and  $V_{*j}$ , and  $S^4$ . The spin cobordism class  $[(F_*, \phi_*)] \in \Omega_2^{\text{spin}} \cong \mathbb{Z}_2$  is the 2-component Sato-Levine invariant of the 2-component link  $(\mathcal{E}_{*i}, \mathcal{E}_{*j})$ .

**Claim 10.2.**  $[(F_0, \phi_0)] = [(F_1, \phi_1)]$ .

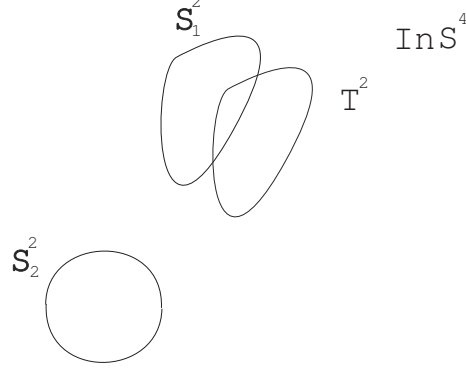


FIGURE 10.1.  $(S_2^2, T^2)$  and  $S_1^2$

**Proof of Claim 10.2.** Since  $p$  is an even number,  $y_i$  and  $y_j$  are odd numbers. By the existence of  $X_i$  and  $X_j$ , we have  $y_i \cdot y_j \cdot [(F_0, \phi_0)] = y_i \cdot y_j \cdot [(F_1, \phi_1)] \in \Omega_2^{\text{spin}}$ . Since  $y_i$  and  $y_j$  are odd numbers,  $[(F_0, \phi_0)] = [(F_1, \phi_1)] \in \Omega_2^{\text{spin}}$ . This completes the proof.  $\square$

Let  $* \in \{0, 1\}$ . Under the above condition, for any distinct  $i, j, k$ , make a 3-component sublink  $Z = (\mathcal{E}_{*i}, \mathcal{E}_{*j}, \mathcal{E}_{*k})$  of  $\mathcal{E}_*$ . Let  $\# = i, j, k$ . Let  $V_{*\#}$  be a special Seifert hypersurface associated with  $Z$ . Let  $C_*$  be the 1-dimensional manifold  $V_{*i} \cap V_{*j} \cap V_{*k}$ . We induce the spin structure  $\tau_*$  on  $C_*$  by using  $V_{*i}$ ,  $V_{*j}$ ,  $V_{*k}$ , and  $S_*^4$ . The spin cobordism class  $[(C_*, \tau_*)] \in \Omega_1^{\text{spin}} \cong \mathbb{Z}_2$  is the 3-component Sato-Levine invariant of the 3-component link  $(\mathcal{E}_{*i}, \mathcal{E}_{*j}, \mathcal{E}_{*k})$ .

**Claim 10.3.**  $[(C_0, \tau_0)] = [(C_1, \tau_1)]$ .

**Proof of Claim 10.3.** By the existence of  $X_i$ ,  $X_j$  and  $X_k$ ,  $y_i \cdot y_j \cdot y_k \cdot [(C_0, \tau_0)] = y_i \cdot y_j \cdot y_k \cdot [(C_1, \tau_1)] \in \Omega_1^{\text{spin}}$ . Since  $y_i$ ,  $y_j$  and  $y_k$  are odd numbers,  $[(C_0, \tau_0)] = [(C_1, \tau_1)]$ .  $\square$

We show an example of two noncobordant surface-links such that we cannot distinguish the nonequivalence of their cobordism classes by using the alinking number associated with a covering-link in the way of Theorem 7.6, but that we can do by using the 2-component Sato-Levine invariant as above. Let the standard  $(S^2, T^2)$ -link be the one of the two surface links. See Figure 10.1. Take a  $(S^2, T^2)$ -link  $(S_1^2, T^2) \subset B^4 \subset S^4$  whose 2-component-Sato-Levine invariant is nonzero. Take the trivial spherical 2-knot  $S_2^2 \subset S^4 - B^4$ . See Figure 10.2. Make a band-sum  $S_1^2 \# T^2$  of  $S_1^2$  and  $T^2$  by using an embedded 3-dimensional 1-handle as drawn in Figure 10.2 so that one of the above  $[(F_*, \phi_*)]$  is nontrivial. Then we obtain a  $(S^2, T^2)$ -link  $(S_2^2, S_1^2 \# T^2)$ . Let this link be the other of the two.

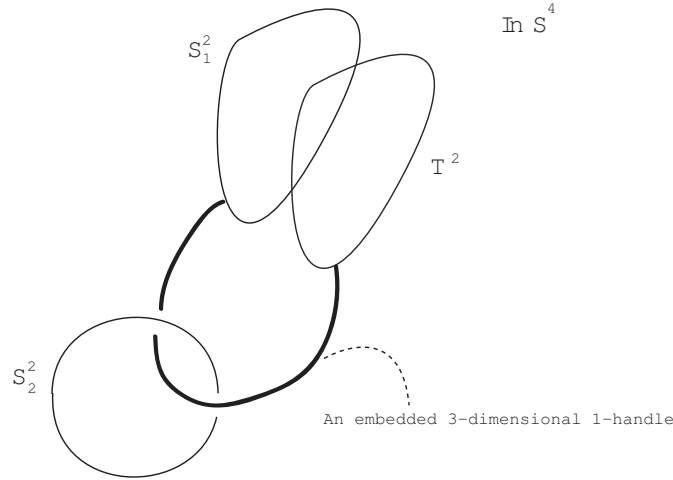


FIGURE 10.2.  $(S_1^2 \# T^2, S_2^2)$

We show an example of two noncobordant surface-links such that we cannot distinguish the nonequivalence of their cobordism classes by using the alinking number associated with a covering link in the way of Theorem 7.6 or by using the 2-component Sato-Levine invariant as above, but that we can do by using the 3-component Sato-Levine invariant as above. Take a 3-component surface-link  $\subset B^4 \subset S^4$  whose 3-component (resp. 2-component)-Sato-Levine invariant is nontrivial (resp. trivial). Take the trivial 2-knot  $\subset S^4 - B^4$ . Make a band-sum by using two (not one) embedded 3-dimensional 1-handles in a similar way to the above one, and obtain a 2-component surface-link  $L$ . Let  $L$  be the one of the two. Let the other of the two the standard surface link which is orientation-preserving diffeomorphic to  $L$ .

Investigate relations between these invariants defined in this section and the other invariants defined or cited in the previous sections.

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